

The generalised scaling function: a systematic study

Davide Fioravanti ^a, Paolo Grinza ^b and Marco Rossi ^{c *}

^a *Sezione INFN di Bologna, Dipartimento di Fisica, Università di Bologna,
Via Irnerio 46, Bologna, Italy*

^b *Departamento de Fisica de Particulas, Universidad de Santiago de Compostela, 15782 Santiago
de Compostela, Spain*

^c *Dipartimento di Fisica dell'Università della Calabria and INFN, Gruppo collegato di Cosenza,
I-87036 Arcavacata di Rende, Cosenza, Italy*

Abstract

We describe a procedure which aims at determining, for all the values of the coupling constant, all the generalised scaling functions arising in the high spin limit for long operators in the $sl(2)$ sector of $\mathcal{N} = 4$ SYM. All the relevant information is encoded in infinite linear systems, each easily connected with the previous ones. Therefore, the solution may be written in a recursive form and then explicitly worked out in the strong coupling regime. In this case, we carefully detail the peculiar convergence of different 'masses' to the unique $O(6)$ mass gap, by also analysing the departure from this limiting theory.

*E-mail: fioravanti@bo.infn.it, pgrinza@usc.es, rossi@cs.infn.it

1 Introduction

One of the most studied sectors of $\mathcal{N} = 4$ SYM is probably the so-called $sl(2)$ sector. In the planar limit, i.e. number of colours $N \rightarrow \infty$ and SYM coupling $g_{YM} \rightarrow 0$ so that the 't Hooft coupling

$$\lambda = 2g_{YM}^2 N = 8\pi^2 g^2 \quad (1.1)$$

stays finite, it results from local composite operators with the form

$$\text{Tr}(\mathcal{D}^s \mathcal{Z}^L) + \dots, \quad (1.2)$$

where \mathcal{D} is the (symmetrised, traceless) covariant derivative acting in all possible ways on the L bosonic fields \mathcal{Z} . The spin of these operators is s and L is the so-called 'twist'. Moreover, this sector would be described – thanks to the AdS/CFT correspondence [1] – by string states on the $\text{AdS}_5 \times S^5$ spacetime with AdS_5 and S^5 charges s and L , respectively. In addition, as far as the one loop is concerned, the Bethe Ansatz problem is equivalent to that of twist operators in QCD [2, 3]; and this partially justifies the great interest in this sector (1.2).

In general, the eigenvectors of the dilatation operator are suitable superpositions of the operators (1.2) with definite anomalous dimension (eigenvalue)

$$\Delta(g, s, L) = L + s + \gamma(g, s, L), \quad (1.3)$$

where $\gamma(g, s, L)$ is the (renormalised) anomalous part. According to the AdS/CFT strong/weak duality, the set of anomalous dimensions of composite operators in $\mathcal{N} = 4$ SYM coincides with the spectrum of energies of the corresponding string states, although the perturbative regimes are interchanged. The highly nontrivial problem of evaluating anomalous dimensions in $\mathcal{N} = 4$ SYM was greatly simplified after the discovery of integrability in the purely bosonic $so(6)$ sector at one loop [4]. Later on, indeed, this fact has been extended to the whole theory and at all loops in the sense that, for instance, any operator of the form (1.2) is associated to one solution of some Bethe Ansatz-like equations and then any anomalous dimension is expressed in terms of one solution [5]. Actually, this is only part of the full story, although the rest should not worry us in this context. In fact, an important limitation emerges as a consequence of the *on-shell* character of the *asymptotic* Bethe Ansatz: as soon as the interaction reaches a range greater than the chain length, then it becomes modified by unpredicted *wrapping effects*. More precisely, the anomalous dimension is *in general* correct up to the $L - 1$ loop in the (SYM) convergent, perturbative expansion, i.e. up to the order g^{2L-2} . Which in particular implies that the asymptotic Bethe Ansatz should give the right result whenever the subsequent limit (1.4) is applied, as in the *leading* expansion (1.5).

The important large twist and high spin limit goes as a double scaling as follows:

$$s \rightarrow \infty, \quad L \rightarrow \infty, \quad j = \frac{L}{\ln s} = \text{fixed}. \quad (1.4)$$

Yet, the string theory becomes dual as semiclassical expansion with the string tension $\sqrt{\lambda} \rightarrow +\infty$ playing the rôle of inverse Planck's constant. This means that this limit is to be considered before

the scaling (1.4) (cf. for instance [6] and references therein), thus imposing a different limit order with respect to our gauge theory approach (cf. below for more details). The relevance of this double limit has been suggested in first instance by [7] within the one-loop SYM theory and then motivated in [6] and in [8] within the string theory dual. In fact, calculations of the latter authors pointed towards the following generalisation (at all loops) for the anomalous dimension formula found in [7]¹

$$\gamma(g, s, L) = f(g, j) \ln s + O((\ln s)^{-\infty}). \quad (1.5)$$

Moreover, by describing the Bethe Ansatz energy through a non-linear integral equation (like in other integrable theories [10]), this Sudakov scaling has been remarkably confirmed in [9]. There this statement was argued by computing iteratively the solution of some integral equations and then, thereof, *the generalised scaling function*, $f(g, j)$ at the first orders in j and g^2 : more precisely the first orders in g^2 have been computed for the first *generalised scaling functions* $f_n(g)$, forming the crucial expansion (see below for the motivation)

$$f(g, j) = \sum_{n=0}^{\infty} f_n(g) j^n. \quad (1.6)$$

As a by-product, the reasonable conjecture has been put forward that the two-variable function $f(g, j)$ should be bi-analytic (in g for fixed j and in j for fixed g). Concentrating on the $f_n(g)$ one easily realizes that in order to check the AdS-CFT correspondence the knowledge of their behaviour for all values of g is extremely important. In particular, comparisons with string theory results involve the strong coupling limit of all the $f_n(g)$'s, after the papers [11, 12, 13] clarified the large g asymptotic expansion of $f_0(g) = f(g)$. In this context, in paper [14] it was shown how to get the contributions beyond the leading scaling function $f(g)$, by means of one linear integral equation, which does not differ from the so-called BES equation, which covers the case $j = 0$, cf. the last of [5], but for the inhomogeneous term. Re-writing this equation in the form of a linear system, in the paper [15] the computation of the first generalised scaling function $f_1(g)$ at strong coupling was performed, showing its proportionality at the leading order to the mass gap $m(g)$ (see (3.13) of the $O(6)$ nonlinear sigma model (NLSM) and giving, consequently, the first evidence of the related proposal² of Alday and Maldacena [8]. Afterwards, the embedding of the $O(6)$ NLSM into $\mathcal{N} = 4$ SYM at large g was shown on formal grounds in [16], where also the leading contribution in the strong coupling limit of $f_3(g)$ was computed. In our contemporaneous paper [17], we could write a set of linear systems of integral equations (of the Fredholm type), whose solutions could give all the generalised scaling functions. The n -th system, which gives the n -th generalised scaling function, needs information involving the solution of the m -th system, with $m \leq n - 3$ and this from the practical point of view allows us an iterative solution. Interestingly, important structures describing the $f_n(g)$ as functions of g started to be identified as the values of the density of Bethe

¹ $O(\ln s^{-\infty})$ means a remainder which goes faster than any power of $\ln s$: $\lim_{s \rightarrow \infty} (\ln s)^k O((\ln s)^{-\infty}) = 0, \forall k > 0$.

²According to this proposal, formulated on the string side of the correspondence, when $g \rightarrow \infty$ and $j/m(g)$ is fixed, $f(g, j) + j$ should coincide with the energy density of the $O(6)$ nonlinear sigma model. Results in the present paper and of [15, 17] concern the case $j \ll m(g)$.

roots and holes and its derivatives at the origin. Explicit evaluations at large g could be done and the strong coupling limit of $f_3(g)$ and $f_4(g)$ was explicitly worked out, finding agreement with analogous results [18] from the $O(6)$ nonlinear sigma model. As stressed in [17], we remark that the agreement for $f_4(g)$ is highly nontrivial, since it involves the full interacting theory of the $O(6)$ NLSM.

In the present paper we want to present a systematic approach which allows us to tackle the computation of all the generalised scaling functions $f_n(g)$ and consequently of $f(g, j)$.

The plan of the paper is as follows. In section 2 we will write general equations for the densities of roots and for the anomalous dimension, which are valid in the regime of large spin s , upon neglecting terms $O((\ln s)^{-\infty})$. In the limit (1.4) such equations will be rewritten (section 3) as an infinite series of linear systems of the Fredholm type, whose solutions give the complete (i.e. all j) generalised scaling function. In Section 4 we will develop a systematic technique which allows to write the solutions of the n -th system - which gives $f_n(g)$ - in terms of the solutions of 'reduced' systems and of the values of the density of Bethe roots and holes and its derivatives in zero. In Section 5 the large coupling limit of all the relevant equations is studied and an explicit, recursive way of finding their solution is given. As an application, in Section 6 the explicit computation of $f_3(g), \dots, f_8(g)$ at strong coupling is performed.

2 High spin equations

In the framework of integrability in $\mathcal{N} = 4$ SYM, it was useful [19] to rewrite the Bethe equations as non-linear integral equations [10]. In particular, this approach was pursued for the $sl(2)$ sector of the theory. In this sector states are described by s Bethe roots, concentrating on an interval $[-b, b]$ of the real line, and L 'holes'. For any state, two holes lie outside the interval $[-b, b]$ and the remaining $L - 2$ holes lie inside this interval. For the lowest anomalous dimension state (ground state) the $L - 2$ internal holes concentrate in the interval $[-c, c]$, $c < b$, and in this interval no roots are present³. The non-linear integral equation for states of the $sl(2)$ sector involves two functions $F(u)$ and $G(u, v)$ satisfying linear equations. It is convenient to split $F(u)$ into its one-loop $F_0(u)$ and higher than one loop $F^H(u)$ contributions and to define the quantities $\sigma_H(u) = \frac{d}{du}F^H(u)$ and $\sigma_0(u) = \frac{d}{du}F_0(u)$, which, as the spin gets large, acquire the meaning of, respectively, higher than one loop and one loop density of both Bethe roots and holes.

In this context, in the large spin limit the ground state anomalous dimension depends on the

³ c is technically known as 'separator'.

densities as follows⁴,

$$\begin{aligned}\gamma(g, s, L) &= -g^2 \int_{-b_0}^{b_0} \frac{dv}{2\pi} \left[\frac{i}{x^+(v)} - \frac{i}{x^-(v)} \right] \sigma_0(v) + \\ &+ g^2 \int_{-\infty}^{+\infty} \frac{dv}{2\pi} \chi_c(v) \left[\frac{i}{x^+(v)} - \frac{i}{x^-(v)} \right] [\sigma_0(v) + \sigma_H(v)] - \\ &- g^2 \int_{-\infty}^{+\infty} \frac{dv}{2\pi} \left[\frac{i}{x^+(v)} - \frac{i}{x^-(v)} \right] \sigma_H(v) + O((\ln s)^{-\infty}),\end{aligned}\quad (2.1)$$

where we introduced the function $\chi_c(u)$ which equals 1 if $-c \leq u \leq c$, where the internal holes concentrate, and 0 otherwise. This means that, as far as the computation of the generalised scaling functions is concerned, one can rely on (2.1). The terms depending on $\sigma_0(u)$ get more manageable after using the important relation introduced in [14],

$$\int_{-b_0}^{b_0} dv f(v) \sigma_0(v) = \int_{-\infty}^{+\infty} dv f(v) \sigma_0^s(v) + O((\ln s)^{-\infty}), \quad (2.2)$$

where the Fourier transform of the function $\sigma_0^s(v)$ satisfies the integral equation,

$$\hat{\sigma}_0^s(k) = -4\pi \frac{\frac{L}{2} - e^{-\frac{|k|}{2}} \cos \frac{ks}{\sqrt{2}}}{2 \sinh \frac{|k|}{2}} - \frac{e^{-\frac{|k|}{2}}}{2 \sinh \frac{|k|}{2}} \int_{-\infty}^{+\infty} du e^{iku} \chi_{c_0}(u) \sigma_0^s(u) - 4\pi \delta(k) \ln 2, \quad (2.3)$$

with the parameter c_0 such that the normalization condition⁵

$$\int_{-\infty}^{+\infty} du \chi_{c_0}(u) \sigma_0^s(u) = -2\pi(L - 2) + O((\ln s)^{-\infty}) \quad (2.4)$$

holds. It is convenient to rewrite the previous relations entirely in terms of Fourier transforms:

$$\begin{aligned}\hat{\sigma}_0^s(k) &= -4\pi \frac{\frac{L}{2} - e^{-\frac{|k|}{2}} \cos \frac{ks}{\sqrt{2}}}{2 \sinh \frac{|k|}{2}} - \\ &- \frac{e^{-\frac{|k|}{2}}}{\sinh \frac{|k|}{2}} \int_{-\infty}^{+\infty} \frac{dh}{2\pi} \hat{\sigma}_0^s(h) \frac{\sin(k-h)c_0}{k-h} - 4\pi \delta(k) \ln 2,\end{aligned}\quad (2.5)$$

$$2 \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \hat{\sigma}_0^s(k) \frac{\sin kc_0}{k} = -2\pi(L - 2) + O((\ln s)^{-\infty}). \quad (2.6)$$

These relations have to be solved together and give, for any values of L , c_0 and $\hat{\sigma}_0^s(k)$ at large spin s . In particular, we can go to the limit (1.4). In this case $\sigma_0^s(u)$ and c_0 expand as,

$$\sigma_0^s(u) = \left[\sum_{n=0}^{\infty} \sigma_0^{s(n)}(u) j^n \right] \ln s + O((\ln s)^{-\infty}), \quad c_0 = \sum_{n=1}^{\infty} c_0^{(n)} j^n, \quad (2.7)$$

⁴The parameter $b_0 > c$ is the one loop contribution to b : therefore, it depends on s through the solution of the linear equation for $F_0(u)$ (see [14]).

⁵The physical meaning of (2.4) is that, for $-c_0 \leq u \leq c_0$, $\sigma_0^s(u)$ approximates the density of holes, that in the one loop theory fill the interval $[-c_0, c_0]$, where no roots are present.

and it is not difficult to give, for instance, the values of the first two coefficients of the expansion of c :

$$c_0^{(1)} = \frac{\pi}{4}, \quad c_0^{(2)} = -\frac{\pi}{4} \ln 2. \quad (2.8)$$

On the other hand from (4.10) of [14] we get that in the large spin limit the higher than one loop density satisfies the linear integral equation⁶,

$$\begin{aligned} \sigma_H(u) = & -iL \frac{d}{du} \ln \left(\frac{1 + \frac{g^2}{2x^-(u)^2}}{1 + \frac{g^2}{2x^+(u)^2}} \right) + \frac{i}{\pi} \int_{-\infty}^{+\infty} dv \chi_c(v) \left[\frac{d}{du} \ln \left(\frac{1 - \frac{g^2}{2x^+(u)x^-(v)}}{1 - \frac{g^2}{2x^-(u)x^+(v)}} \right) + \right. \\ & + i \frac{d}{du} \theta(u, v) + i \frac{1}{1 + (u - v)^2} \left. \right] [\sigma_0^s(v) + \sigma_H(v)] \\ & - \frac{i}{\pi} \int_{-\infty}^{+\infty} dv \frac{d}{du} \left[\ln \left(\frac{1 - \frac{g^2}{2x^+(u)x^-(v)}}{1 - \frac{g^2}{2x^-(u)x^+(v)}} \right) + i \theta(u, v) \right] \sigma_0^s(v) + \\ & + \int_{-\infty}^{+\infty} \frac{dv}{\pi} \frac{1}{1 + (u - v)^2} \sigma_H(v) + \int_{-\infty}^{+\infty} \frac{dv}{\pi} \chi_{c_0}(v) \frac{1}{1 + (u - v)^2} \sigma_0^s(v) - \\ & - \frac{i}{\pi} \int_{-\infty}^{+\infty} dv \frac{d}{du} \left[\ln \left(\frac{1 - \frac{g^2}{2x^+(u)x^-(v)}}{1 - \frac{g^2}{2x^-(u)x^+(v)}} \right) + i \theta(u, v) \right] \sigma_H(v) + O((\ln s)^{-\infty}), \end{aligned} \quad (2.9)$$

which has to be solved together with the conditions,

$$\begin{aligned} \int_{-\infty}^{+\infty} du \chi_c(u) [\sigma_0^s(u) + \sigma_H(u)] &= -2\pi(L - 2) + O((\ln s)^{-\infty}), \\ \int_{-\infty}^{+\infty} du \chi_{c_0}(u) \sigma_0^s(u) &= -2\pi(L - 2) + O((\ln s)^{-\infty}). \end{aligned} \quad (2.10)$$

As in the one loop case, it is convenient to rewrite, in terms of Fourier transforms, equation (2.9),

$$\begin{aligned} \hat{\sigma}_H(k) = & \pi L \frac{1 - J_0(\sqrt{2}gk)}{\sinh \frac{|k|}{2}} + \\ & + \frac{1}{\sinh \frac{|k|}{2}} \int_{-\infty}^{+\infty} \frac{dh}{|h|} \left[\sum_{r=1}^{\infty} r(-1)^{r+1} J_r(\sqrt{2}gk) J_r(\sqrt{2}gh) \frac{1 - \text{sgn}(kh)}{2} e^{-\frac{|h|}{2}} + \right. \\ & + \text{sgn}(h) \sum_{r=2}^{\infty} \sum_{\nu=0}^{\infty} c_{r,r+1+2\nu} (-1)^{r+\nu} e^{-\frac{|h|}{2}} \left(J_{r-1}(\sqrt{2}gk) J_{r+2\nu}(\sqrt{2}gh) - \right. \\ & - \left. J_{r-1}(\sqrt{2}gh) J_{r+2\nu}(\sqrt{2}gk) \right) \left. \right] \left[\hat{\sigma}_0^s(h) + \hat{\sigma}_H(h) - \int_{-\infty}^{+\infty} \frac{dp}{2\pi} (\hat{\sigma}_0^s(p) + \hat{\sigma}_H(p)) 2 \frac{\sin(h-p)c}{h-p} \right] - \\ & - \frac{e^{-\frac{|k|}{2}}}{\sinh \frac{|k|}{2}} \int_{-\infty}^{+\infty} \frac{dp}{2\pi} (\hat{\sigma}_0^s(p) + \hat{\sigma}_H(p)) \frac{\sin(k-p)c}{k-p} + \frac{e^{-\frac{|k|}{2}}}{\sinh \frac{|k|}{2}} \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \hat{\sigma}_0^s(p) \frac{\sin(k-p)c_0}{k-p} + \\ & + O((\ln s)^{-\infty}), \end{aligned} \quad (2.11)$$

⁶The quantity $\theta(u, v)$ appearing in (2.9) is the well known dressing factor. For related notations, we refer to the last of [5].

and also the normalization conditions,

$$2 \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \hat{\sigma}_0^s(k) \frac{\sin kc_0}{k} = -2\pi(L-2) + O((\ln s)^{-\infty}), \quad (2.12)$$

$$2 \int_{-\infty}^{+\infty} \frac{dk}{2\pi} [\hat{\sigma}_0^s(k) + \hat{\sigma}_H(k)] \frac{\sin kc}{k} = -2\pi(L-2) + O((\ln s)^{-\infty}).$$

Again, in the limit (1.4) both c and $\sigma_H(u)$ expand in powers of j ,

$$\sigma_H(u) = \left[\sum_{n=0}^{\infty} \sigma_H^{(n)}(u) j^n \right] \ln s + O((\ln s)^{-\infty}), \quad c = \sum_{n=1}^{\infty} c^{(n)} j^n, \quad (2.13)$$

so that we find convenient to use the all loops density⁷

$$\sigma(u) = \sigma_H(u) + \sigma_0^s(u), \quad (2.14)$$

which, in the limit (1.4) expands as

$$\sigma(u) = \left[\sum_{n=0}^{\infty} \sigma^{(n)}(u) j^n \right] \ln s + O((\ln s)^{-\infty}). \quad (2.15)$$

As in the one loop case, it is easy to give the first orders in the expansion of c in the limit (1.4):

$$c^{(1)} = \frac{\pi}{4 - \sigma_H^{(0)}(0)}, \quad c^{(2)} = -\pi \frac{4 \ln 2 - \sigma_H^{(1)}(0)}{[4 - \sigma_H^{(0)}(0)]^2}. \quad (2.16)$$

Finally, by comparing (2.1) with (2.11), we realize that, very easily, when $s \rightarrow \infty$,

$$\gamma(g, s, L) = \frac{1}{\pi} \lim_{k \rightarrow 0} \hat{\sigma}_H(k) + O((\ln s)^{-\infty}). \quad (2.17)$$

And this relation extends the validity of the analogue in [20], valid only at order $\ln s$, to all orders $(\ln s)^{-m}$, $m \in \mathbb{N}$. Equality (2.17) implies, very simply, that

$$f_n(g) = \frac{1}{\pi} \hat{\sigma}_H^{(n)}(0). \quad (2.18)$$

A systematic approach to the computation of $f_n(g)$ using the solution of (2.11) and explicit formulæ which allow their exact determination at strong coupling is the main issue and the main result of this paper.

⁷The one loop quantity $\sigma_0^s(u)$ is an approximation, according to (2.2), to the real one loop density. Since in this article we will always neglect terms $O((\ln s)^{-\infty})$, such approximation is completely justified.

3 On the calculation of the generalized scaling functions

From result (2.18) we realize that the generalised scaling functions $f_n(g)$ can be extracted from the n -th component $\sigma_H^{(n)}(u)$ of the solution of (2.9) in the limit (1.4). We are therefore going to analyze in a systematic way (2.9) or (2.11) in such a limit. Equations for $\sigma_H^{(0)}(u)$ and $\sigma_H^{(1)}(u)$ were already studied: the former is the well known BES equation (last of [5]), the latter was treated in detail in [15]. Since in this paper we will use results involving $\sigma_H^{(0)}(u)$ and $\sigma_H^{(1)}(u)$, we will treat also briefly these two cases, which, in addition, need to be considered separately from the rest of the $\sigma_H^{(n)}(u)$.

3.1 The BES equation

If we restrict (2.11) to the component proportional to $\ln s$ we get the BES equation for $\hat{\sigma}_H^{(0)}(k)$. We now briefly describe how to rewrite such equation in a form suitable for our future manipulations. We first define

$$S^{(0)}(k) = \frac{2 \sinh \frac{|k|}{2}}{2\pi|k|} \hat{\sigma}_H^{(0)}(k), \quad (3.1)$$

and then expand $S^{(0)}(k)$, $k \geq 0$, in series of Bessel functions,

$$S^{(0)}(k) = \sum_{p=1}^{\infty} S_{2p}^{(0)}(g) \frac{J_{2p}(\sqrt{2}gk)}{k} + \sum_{p=1}^{\infty} S_{2p-1}^{(0)}(g) \frac{J_{2p-1}(\sqrt{2}gk)}{k}. \quad (3.2)$$

On the coefficients $S_p^{(0)}(g)$ the BES equation implies the linear system,

$$\begin{aligned} S_{2p}^{(0)}(g) &= -4p \sum_{m=1}^{\infty} Z_{2p,2m}(g) S_{2m}^{(0)}(g) + 4p \sum_{m=1}^{\infty} Z_{2p,2m-1}(g) S_{2m-1}^{(0)}(g) \\ S_{2p-1}^{(0)}(g) &= 2\sqrt{2}g \delta_{p,1} - 2(2p-1) \sum_{m=1}^{\infty} Z_{2p-1,2m}(g) S_{2m}^{(0)}(g) - 2(2p-1) \sum_{m=1}^{\infty} Z_{2p-1,2m-1}(g) S_{2m-1}^{(0)}(g), \end{aligned} \quad (3.3)$$

where we introduced the notation

$$Z_{n,m}(g) = \int_0^{+\infty} \frac{dh}{h} \frac{J_n(\sqrt{2}gh) J_m(\sqrt{2}gh)}{e^h - 1}. \quad (3.4)$$

The cusp anomalous dimension can be extracted from the relations

$$\lim_{k \rightarrow 0^+} S^{(0)}(k) = \frac{1}{2} f(g), \quad f(g) = \sqrt{2}g S_1^{(0)}(g), \quad (3.5)$$

and its strong coupling behaviour was completely disentangled in [12].

3.2 On the first generalized scaling function

The first generalised scaling function was studied in [15]. Here we briefly recall the main results. We define the even function

$$S^{(1)}(k) = \frac{2 \sinh \frac{|k|}{2}}{2\pi|k|} \hat{\sigma}_H^{(1)}(k), \quad (3.6)$$

and introduce the two functions

$$a_r(g) = \int_{-\infty}^{+\infty} \frac{dh}{h} J_r(\sqrt{2}gh) \frac{1}{1 + e^{\frac{|h|}{2}}}, \quad \bar{a}_r(g) = \int_{-\infty}^{+\infty} \frac{dh}{|h|} J_r(\sqrt{2}gh) \frac{1}{1 + e^{\frac{|h|}{2}}}. \quad (3.7)$$

Expanding, for $k \geq 0$, in a series involving Bessel functions,

$$S^{(1)}(k) = \sum_{p=1}^{\infty} S_{2p}^{(1)}(g) \frac{J_{2p}(\sqrt{2}gk)}{k} + \sum_{p=1}^{\infty} S_{2p-1}^{(1)}(g) \frac{J_{2p-1}(\sqrt{2}gk)}{k}, \quad (3.8)$$

the coefficients $S_r^{(1)}(g)$ satisfy the linear system,

$$S_{2p}^{(1)}(g) = 2 + 2p \left(-\bar{a}_{2p}(g) - 2 \sum_{m=1}^{\infty} Z_{2p,2m}(g) S_{2m}^{(1)}(g) + 2 \sum_{m=1}^{\infty} Z_{2p,2m-1}(g) S_{2m-1}^{(1)}(g) \right), \quad (3.9)$$

$$\frac{S_{2p-1}^{(1)}(g)}{2p-1} = -a_{2p-1}(g) - 2 \sum_{m=1}^{\infty} Z_{2p-1,2m}(g) S_{2m}^{(1)}(g) - 2 \sum_{m=1}^{\infty} Z_{2p-1,2m-1}(g) S_{2m-1}^{(1)}(g).$$

In paper [15] we found the following asymptotic strong coupling solution to the system (3.9):

$$S_{2m-1}^{(1)}(g) = (2m-1) \sum_{n'=1}^m (-1)^{n'} \frac{\Gamma(m+n'-1)}{\Gamma(m-n'+1)} \frac{b_{2n'-1}}{g^{2n'-1}}, \quad (3.10)$$

$$S_{2m}^{(1)}(g) = -2m \sum_{n'=1}^m (-1)^{n'} \frac{\Gamma(m+n')}{\Gamma(m-n'+1)} \frac{b_{2n'}}{g^{2n'}},$$

where the coefficients

$$b_{2n'} = 2^{-n'} (-1)^{n'} \sum_{k=0}^{n'} \frac{E_{2k} 2^{2k}}{(2k)!(2n'-2k)!}, \quad (3.11)$$

$$b_{2n'-1} = 2^{-n'+\frac{1}{2}} (-1)^{n'-1} \sum_{k=0}^{n'-1} \frac{E_{2k} 2^{2k}}{(2k)!(2n'-2k-1)!},$$

with E_{2k} the Euler's numbers, sum up to the generating function

$$b(t) = \sum_{n'=0}^{\infty} b_{n'} t^{n'} = \frac{1}{\cos \frac{t}{\sqrt{2}} - \sin \frac{t}{\sqrt{2}}}. \quad (3.12)$$

In addition, the behaviour

$$f_1(g) = 2 \lim_{k \rightarrow 0^+} S^{(1)}(k) = \sqrt{2}gS_1^{(1)}(g) = -1 + m(g), \quad m(g) = \frac{2^{\frac{5}{8}}\pi}{\Gamma(\frac{5}{4})}g^{\frac{1}{4}}e^{-\frac{\pi}{\sqrt{2}}g} [1 + O(1/g)], \quad (3.13)$$

where $m(g)$ agrees at the leading order with the mass gap of the $O(6)$ nonlinear sigma model, expressed in terms of parameters of the underlying $\text{AdS}_5 \times S^5$ sigma model, was shown after numerically solving the system (3.9).

3.3 On the second and higher generalized scaling functions

We now give a general scheme for tackling the problem of computing the n -th generalised scaling function $f_n(g)$ for $n \geq 2$ at arbitrary value of the coupling constant.

We start from (2.11) and define the function $S(k)$:

$$\ln s \ S(k) = \frac{2 \sinh \frac{|k|}{2}}{2\pi|k|} [\hat{\sigma}_H(k) + \hat{\sigma}_0^s(k)] + \frac{e^{-\frac{|k|}{2}}}{\pi|k|} \int_{-\infty}^{+\infty} \frac{dp}{2\pi} [\hat{\sigma}_0^s(p) + \hat{\sigma}_H(p)] \frac{\sin(k-p)c}{k-p}, \quad (3.14)$$

which, differently from the cases $n = 0$, $n = 1$, depends on the all loops density (2.14) $\hat{\sigma}(k) = \hat{\sigma}_H(k) + \hat{\sigma}_0^s(k)$. We go to the limit (1.4). We have

$$S(k) = s^{(0)}(k) + s^{(1)}(k)j + \sum_{n=2}^{\infty} S^{(n)}(k)j^n. \quad (3.15)$$

and we concentrate on $S^{(n)}(k)$, with $n \geq 2$. For such functions the following equation hold:

$$\begin{aligned} S^{(n)}(k) = & \frac{1}{\pi|k|} \int_{-\infty}^{+\infty} \frac{dh}{|h|} \left[\sum_{r=1}^{\infty} r(-1)^{r+1} J_r(\sqrt{2}gk) J_r(\sqrt{2}gh) \frac{1 - \text{sgn}(kh)}{2} e^{-\frac{|h|}{2}} + \right. \\ & + \text{sgn}(h) \sum_{r=2}^{\infty} \sum_{\nu=0}^{\infty} c_{r,r+1+2\nu} (-1)^{r+\nu} e^{-\frac{|h|}{2}} \left(J_{r-1}(\sqrt{2}gk) J_{r+2\nu}(\sqrt{2}gh) - \right. \\ & \left. \left. - J_{r-1}(\sqrt{2}gh) J_{r+2\nu}(\sqrt{2}gk) \right) \right] \left[\frac{\pi|h|}{\sinh \frac{|h|}{2}} S^{(n)}(h) - \frac{e^{\frac{|h|}{2}}}{\sinh \frac{|h|}{2}} \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \hat{\sigma}(p) \frac{\sin(h-p)c}{h-p} \right]_{j^n}, \end{aligned} \quad (3.16)$$

where the simbol $|_{j^n}$ means that we have to extract only the coefficient of j^n in the limit (1.4), after having removed the overall factor $\ln s$.

Again, if we restrict the domain to $k \geq 0$ we can expand in series of Bessel functions,

$$S^{(n)}(k) = \sum_{p=1}^{\infty} S_{2p}^{(n)}(g) \frac{J_{2p}(\sqrt{2}gk)}{k} + \sum_{p=1}^{\infty} S_{2p-1}^{(n)}(g) \frac{J_{2p-1}(\sqrt{2}gk)}{k}, \quad (3.17)$$

in such a way that the n -th generalised scaling function is expressed as (2.17,3.14):

$$f_n(g) = \sqrt{2}gS_1^{(n)}(g). \quad (3.18)$$

After some computations, we find the following system of equations for the coefficients of $S^{(n)}(k)$, with $n \geq 2$,

$$\begin{aligned} S_{2p}^{(n)}(g) &= A_{2p}^{(n)}(g) - 4p \sum_{m=1}^{\infty} Z_{2p,2m}(g) S_{2m}^{(n)}(g) + 4p \sum_{m=1}^{\infty} Z_{2p,2m-1}(g) S_{2m-1}^{(n)}(g), \\ S_{2p-1}^{(n)}(g) &= A_{2p-1}^{(n)}(g) - 2(2p-1) \sum_{m=1}^{\infty} Z_{2p-1,2m}(g) S_{2m}^{(n)}(g) - \\ &\quad - 2(2p-1) \sum_{m=1}^{\infty} Z_{2p-1,2m-1}(g) S_{2m-1}^{(n)}(g), \end{aligned} \quad (3.19)$$

where the 'forcing terms' $A_r^{(n)}(g)$ are given by:

$$A_r^{(n)}(g) = r \int_0^{+\infty} \frac{dh}{2\pi h} \frac{J_r(\sqrt{2gh})}{\sinh \frac{h}{2}} \int_{-\infty}^{+\infty} \frac{dp}{2\pi} 2 \frac{\sin((h-p)c)}{h-p} [\hat{\sigma}_0^s(p) + \hat{\sigma}_H(p)] \Big|_{j^n}. \quad (3.20)$$

These systems have all the same kernel, which coincides with the BES one, and differ only for their forcing terms. The enforcing of the normalization conditions in (3.20) will show how the n -th forcing term depend on the solutions of the m -th system, with $m \leq n-3$, allowing, therefore, their iterative solution. This will be the subject of next section, where we are going to systematically tackle the problem of finding $A_r^{(n)}(g)$ for all values of n , up to the desired order.

4 Systematorics

The main obstacle to obtain a fully explicit expression for the infinite linear system at a generic order n is the double expansion contained in the term $\sin((h-p)c(j))$ of equation (3.20). A similar structure is also present in the normalization conditions (2.12).

In order to overcome this technical problem, it is worth to remember a standard result of combinatorics known as the Faà di Bruno's formula [21]. Let $f(x)$ and $g(x)$ be a pair of functions admitting (at least formally) a power expansion of this kind

$$f(x) = \sum_{n=1}^{\infty} \frac{f_n}{n!} x^n, \quad g(x) = \sum_{n=1}^{\infty} \frac{g_n}{n!} x^n, \quad (4.1)$$

then the composition $g(f(x))$ admits the following power expansion

$$g(f(x)) = h(x) = \sum_{n=1}^{\infty} \frac{h_n}{n!} x^n, \quad (4.2)$$

where the coefficients h_n have the following form

$$h_n = \sum_{k=1}^n g_k B_{n,k}(f_1, \dots, f_{n-k+1}). \quad (4.3)$$

$B_{n,k}(\underline{f})$ is the Bell polynomial defined as

$$B_{n,k}(f_1, \dots, f_{n-k+1}) = n! \sum_{\{j_1, \dots, j_{n-k+1}\}} \prod_{m=1}^{n-k+1} \frac{(f_m)^{j_m}}{j_m! (m!)^{j_m}} \quad (4.4)$$

and the sum runs over all the non negative j 's satisfying the conditions

$$\sum_{m=1}^{n-k+1} j_m = k, \quad \sum_{m=1}^{n-k+1} m j_m = n.$$

The previous equation will be our main tool in the remaining part of the section. It is straightforward to apply the previous formula to the present case $\sin(p c(j))$ with

$$\sin x = \sum_{n=1}^{\infty} \frac{\xi_n}{n!} x^n, \quad \xi_n = \frac{1}{2} i^{n+1} ((-1)^n - 1) \quad (4.5)$$

and

$$c(j) = \sum_{n=1}^{\infty} c^{(n)} j^n. \quad (4.6)$$

We end up with (we divide by p for future convenience)

$$\begin{aligned} \frac{\sin(p c(j))}{p} &= \sum_{n=1}^{\infty} \frac{\sum_{k=1}^n \xi_k}{p n!} B_{n,k}(p c^{(1)}, \dots, p (n-k+1)! c^{(n-k+1)}) j^n = \\ &= \sum_{n=1}^{\infty} \Lambda_n(p) j^n. \end{aligned} \quad (4.7)$$

Let us now use this result in order to write in a more convenient way both the normalization condition (2.12) and the forcing term (3.20). We begin with the analysis of $\Lambda_n(p)$ in order to put it in a more suitable form. Some elementary manipulations bring it to the form

$$\Lambda_n(p) = \sum_{k=1}^n \xi_k \beta_{n,k}(c^{(1)}, \dots, c^{(n-k+1)}; p), \quad (4.8)$$

with

$$\begin{aligned} \beta_{n,k}(c^{(1)}, \dots, c^{(n-k+1)}; p) &= \sum_{\{j_1, \dots, j_{n-k+1}\}} (p^{k-1}) \prod_{m=1}^{n-k+1} \frac{(c^{(m)})^{j_m}}{j_m!}, \\ \sum_{m=1}^{n-k+1} j_m &= k, \quad \sum_{m=1}^{n-k+1} m j_m = n. \end{aligned} \quad (4.9)$$

The forcing term and the normalization conditions have a common structure

$$\begin{aligned} \frac{\sin(p_1 c(j))}{p_1} \hat{\sigma}(p_2) &= \left(\sum_{n=1}^{\infty} \Lambda_n(p_1) j^n \right) \left(\sum_{n=0}^{\infty} \hat{\sigma}^{(n)}(p_2) j^n \right) \ln s = \\ &= \sum_{n=1}^{\infty} \Gamma_n(p_1, p_2) j^n \ln s, \end{aligned} \quad (4.10)$$

where

$$\Gamma_n(p_1, p_2) = \sum_{k=1}^n \Lambda_k(p_1) \hat{\sigma}^{(n-k)}(p_2). \quad (4.11)$$

We can finally write the coefficient in the expansion in powers of j of the integral over the momentum which appears in the forcing term as

$$2 \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \Gamma_n(h-p, p) \ln s, \quad (4.12)$$

along with the normalization condition

$$\int_{-\infty}^{+\infty} \frac{dp}{2\pi} \Gamma_n(p, p) = -\pi \delta_{n,1}. \quad (4.13)$$

Our next step will be to enforce the normalization condition in the forcing term in order to gain a simplification of its structure. We notice that $\Lambda_n(p)$ has a momentum independent term corresponding to the term $k = 1$ in the sum (4.8) and hence $\Gamma_n(p_1, p_2)$ admits the following decomposition

$$\Gamma_n(p_1, p_2) = \Gamma_n^{(0)}(p_2) + \tilde{\Gamma}_n(p_1, p_2), \quad \Gamma_n^{(0)}(p_2) = \sum_{k=1}^n \hat{\sigma}^{(n-k)}(p_2) c^{(k)}. \quad (4.14)$$

The normalization condition then becomes

$$- \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \Gamma_n^{(0)}(p) = \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \tilde{\Gamma}_n(p, p) + \pi \delta_{n,1}, \quad (4.15)$$

which allows to subtract the $\Gamma_n^{(0)}(p)$ contribution in the forcing term:

$$\begin{aligned} 2 \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \Gamma_n(h-p, p) &= \\ &= -2\pi \delta_{n,1} + 2 \int_{-\infty}^{+\infty} \frac{dp}{2\pi} [\Gamma_n(h-p, p) - \Gamma_n(p, p)] \\ &= -2\pi \delta_{n,1} + 2 \int_{-\infty}^{+\infty} \frac{dp}{2\pi} [\Delta_n(h-p, p)]. \end{aligned} \quad (4.16)$$

A brief inspection allows to realize that such a subtraction is responsible for the fact that $A_r^{(2)} = 0$ which imply $f_2(g) = 0$, as already noticed in the literature.

The final step is to make such a subtraction explicit for any n . In order to achieve this, we have to take into account the properties of parity of the Fourier transform of the densities of Bethe roots (they are even with respect to the momentum p), and the following relation which allows to perform the integral over p ,

$$i^{-s} d_s \sigma^{(n-k);(s)} = i^{-s} \frac{d^s \sigma^{(n-k)}(u)}{du^s} \Big|_{u=0} = \int_{-\infty}^{+\infty} \frac{dp}{2\pi} p^s \hat{\sigma}^{(n-k)}(p), \quad d_s = \frac{1}{2}(1 + (-1)^s), \quad (4.17)$$

which is different from zero only for s even, due to the parity properties of $\hat{\sigma}^{(n-k)}(p)$.

It is then possible to rewrite $\Delta_n(h-p, p)$ as follows

$$\Delta_n(h-p, p) = \sum_{k=1}^n \hat{\sigma}^{(n-k)}(p) (\Lambda_k(h-p) - \Lambda_k(p)), \quad (4.18)$$

where

$$\begin{aligned} \Lambda_k(h-p) - \Lambda_k(p) &= \sum_{l=1}^k \xi_l \sum_{\{j_1, \dots, j_{k-l+1}\}} \prod_{m=1}^{k-l+1} \frac{(c^{(m)})^{j_m}}{j_m!} ((h-p)^{l-1} - p^{l-1}) \\ &= \sum_{l=1}^k \xi_l \sum_{\{j_1, \dots, j_{k-l+1}\}} \prod_{m=1}^{k-l+1} \frac{(c^{(m)})^{j_m}}{j_m!} \left(\sum_{s=0}^{l-1} \binom{l-1}{s} h^{l-1-s} (-1)^s p^s - p^{l-1} \right) \\ &= \sum_{l=3}^k \xi_l \sum_{\{j_1, \dots, j_{k-l+1}\}} \prod_{m=1}^{k-l+1} \frac{(c^{(m)})^{j_m}}{j_m!} \left(\sum_{s=0}^{l-2} \binom{l-1}{s} h^{l-1-s} (-1)^s p^s + (-p)^{l-1} - p^{l-1} \right) \\ &= \sum_{l=3}^k \xi_l \left(\sum_{s=0}^{l-2} \binom{l-1}{s} h^{l-1-s} (-p)^s \right) \sum_{\{j_1, \dots, j_{k-l+1}\}} \prod_{m=1}^{k-l+1} \frac{(c^{(m)})^{j_m}}{j_m!}, \quad (4.19) \\ \sum_{m=1}^{k-l+1} j_m &= l, \quad \sum_{m=1}^{k-l+1} m j_m = k. \end{aligned}$$

The last step comes from the fact that it is always $(-p)^{l-1} - p^{l-1} = 0$, because ξ_l is non-vanishing only for odd l . One can also notice that the subtraction and the fact that $\xi_2 = 0$ allow the sum over l to begin from $l = 3$.

The previous result, together with eq. (4.17), allows to write down, for $n \geq 2$,

$$\begin{aligned} 2 \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \Gamma_n(h-p, p) &= 2 \int_{-\infty}^{+\infty} \frac{dp}{2\pi} [\Gamma_n(h-p, p) - \Gamma_n(p, p)] \quad (4.20) \\ &= 2 \int_{-\infty}^{+\infty} \frac{dp}{2\pi} [\Delta_n(h-p, p)] = \\ &= 2 \sum_{k=1}^n \sum_{l=3}^k \xi_l \left(\sum_{s=0}^{l-2} \binom{l-1}{s} d_s h^{l-1-s} (-i)^{-s} \sigma^{(n-k);(s)} \right) \sum_{\{j_1, \dots, j_{k-l+1}\}} \prod_{m=1}^{k-l+1} \frac{(c^{(m)})^{j_m}}{j_m!}, \\ \sum_{m=1}^{k-l+1} j_m &= l, \quad \sum_{m=1}^{k-l+1} m j_m = k, \end{aligned}$$

which is nothing but the explicit n -th term of the j expansion of the integral over p which appears in the forcing term. Then, if we pose

$$\mathbb{I}_r^{l,s} = r \int_0^{+\infty} \frac{dh}{2\pi h} \frac{J_r(\sqrt{2gh})}{\sinh \frac{h}{2}} h^{l-1-s},$$

we can explicitly write down the generic expression for the forcing term $A_r^{(n)}$, $n \geq 2$, entering the system (3.19) for the n -th term of the j expansion of the function $S(k)$:

$$\begin{aligned}
A_r^{(n)}(g) &= \\
&= 2 \sum_{k=1}^n \sum_{l=3}^k \xi_l \left(\sum_{s=0}^{l-2} \binom{l-1}{s} d_s (-i)^{-s} \sigma^{(n-k);(s)} \mathbb{I}_r^{l,s} \right) \sum_{\{j_1, \dots, j_{k-l+1}\}} \prod_{m=1}^{k-l+1} \frac{(c^{(m)})^{j_m}}{j_m!}, \\
&\sum_{m=1}^{k-l+1} j_m = l, \quad \sum_{m=1}^{k-l+1} m j_m = k.
\end{aligned} \tag{4.21}$$

Because of the particular form of the forcing terms, it is convenient to write the solution of (3.19) as

$$\begin{aligned}
S_r^{(n)}(g) &= \\
&= 2 \sum_{k=1}^n \sum_{l=3}^k \xi_l \left(\sum_{s=0}^{l-2} \binom{l-1}{s} d_s (-i)^{-s} \sigma^{(n-k);(s)} \tilde{S}_r^{(\frac{l-s-1}{2})}(g) \right) \sum_{\{j_1, \dots, j_{k-l+1}\}} \prod_{m=1}^{k-l+1} \frac{(c^{(m)})^{j_m}}{j_m!}, \\
&\sum_{m=1}^{k-l+1} j_m = l, \quad \sum_{m=1}^{k-l+1} m j_m = k,
\end{aligned} \tag{4.22}$$

where the “reduced” coefficients $\tilde{S}_r^{(k)}$ satisfy the equations

$$\begin{aligned}
\tilde{S}_{2p}^{(k)}(g) &= \mathbb{I}_{2p}^{(k)}(g) - 4p \sum_{m=1}^{\infty} Z_{2p,2m}(g) \tilde{S}_{2m}^{(k)}(g) + 4p \sum_{m=1}^{\infty} Z_{2p,2m-1}(g) \tilde{S}_{2m-1}^{(k)}(g) \\
\tilde{S}_{2p-1}^{(k)}(g) &= \mathbb{I}_{2p-1}^{(k)}(g) - 2(2p-1) \sum_{m=1}^{\infty} Z_{2p-1,2m}(g) \tilde{S}_{2m}^{(k)}(g) - \\
&- 2(2p-1) \sum_{m=1}^{\infty} Z_{2p-1,2m-1}(g) \tilde{S}_{2m-1}^{(k)}(g),
\end{aligned} \tag{4.23}$$

with the reduced forcing terms,

$$\mathbb{I}_r^{(k)} = r \int_0^{+\infty} \frac{dh}{2\pi} h^{2k-1} \frac{J_r(\sqrt{2}gh)}{\sinh \frac{h}{2}}, \tag{4.24}$$

which are ‘known’ functions, i.e. they do not depend on the quantities $\sigma^{(n');(s)}$. We notice that inside the structure of the forcing term $A_r^{(n)}(g)$ (4.21) we find the constants $c^{(m)}$, with $m \leq n-2$ and the densities of the Bethe roots at $u=0$ (together with their derivatives) $\sigma^{(n');(s)}$, with $n' \leq n-3$. In addition, the constants $c^{(m)}$ can be related to $\sigma^{(n');(s)}$, with $n' \leq m-1$, by means of the normalization condition (4.13), thus leaving the forcing term $A_r^{(n)}(g)$ as dependent on $\sigma^{(n');(s)}$, with $n' \leq n-3$. Let us now show this. We first of all notice that, for $n=1$, we have

$$c^{(1)} = -\frac{\pi}{\sigma^{(0);(0)}} \tag{4.25}$$

and that, for $m > 1$, the normalization condition takes the form

$$\begin{aligned} \sum_{k=1}^m \sum_{l=1}^k \xi_l (i)^{-l+1} \sigma^{(m-k);(l-1)} \sum_{\{j_1, \dots, j_{k-l+1}\}} \prod_{m'=1}^{k-l+1} \frac{(c^{(m')})^{j_{m'}}}{j_{m'}!} &= 0, \\ \sum_{m'=1}^{k-l+1} j_{m'} &= l, \quad \sum_{m'=1}^{k-l+1} m' j_{m'} = k. \end{aligned} \quad (4.26)$$

After a brief inspection of the latter it is possible to realize that, at order m , the only term which contains $c^{(m)}$ can be singled out by taking $k = m$, $l = 1$, and that the remaining terms in the sums only contain $c^{(k)}$ with $k < m$.

As a consequence, we can write a recurrence relation

$$\begin{aligned} c^{(m)} &= - \sum_{k=1}^{m-1} \frac{\sigma^{(m-k);(0)}}{\sigma^{(0);(0)}} c^{(k)} - \\ &- \sum_{k=1}^m \sum_{l=2}^k \xi_l (i)^{-l+1} \frac{\sigma^{(m-k);(l-1)}}{\sigma^{(0);(0)}} \sum_{\{j_1, \dots, j_{k-l+1}\}} \prod_{m'=1}^{k-l+1} \frac{(c^{(m')})^{j_{m'}}}{j_{m'}!}, \quad m > 1, \\ \sum_{m'=1}^{k-l+1} j_{m'} &= l, \quad \sum_{m'=1}^{k-l+1} m' j_{m'} = k. \end{aligned} \quad (4.27)$$

which, together with the initial condition (4.25), allows to express all the $c^{(m)}$ recursively, in terms of $\sigma^{(n');(s)}$, with $n' \leq m-1$. Therefore, we conclude that the forcing term $A_r^{(n)}(g)$ (4.21) and the solution $S_r^{(n)}(g)$ (4.22) actually depend only on $\sigma^{(n');(s)}$, with $n' \leq n-3$, i.e. on the solutions of previous systems. Consequently, at least in principle, the solution for the $S_r^{(n)}$ may be written in a recursive form.

To summarize, the principal result of this section is formula (4.22): the evaluation of the n -th generalised scaling function $f_n(g) = \sqrt{2}gS_1^{(n)}(g)$, for $n \geq 2$, is eventually reduced to the knowledge of $\tilde{S}_1^{(k)}(g)$ and of the densities and their derivatives in zero, $\sigma^{(n');(s)}$ (4.17), with $n' \leq n-3$. In next subsection, we will show that $\tilde{S}_1^{(k)}(g)$ (and $f_1(g) = \sqrt{2}gS_1^{(1)}(g)$) can be given an integral representation in terms of the solution of the BES equation. However, this connection to the BES equation (true, for obvious reasons, also for $\sigma^{(0);(s)}$) is not true for the densities and their derivatives at zero $\sigma^{(n');(s)}$, $n' \geq 1$: in order to find them, one needs more additional information, i.e. the full solution $S_r^{(1)}(g)$, $\tilde{S}_r^{(k)}(g)$, for all r , to the systems (3.9), (4.23). However, we again stress that, due to iterative structure of (4.22), an explicit solution for the $S_r^{(n)}(g)$ can be found by recursive methods. This will be explicitly shown in the strong coupling limit (Section 5).

4.1 Mapping the reduced systems to the BES equation

As stated before, the main point of this subsection is to write down an integral representation for the reduced coefficient $\tilde{S}_1^{(k)}(g)$ and for $S_1^{(1)}(g)$, in terms of the solution of the BES equation.

As a first step we rewrite the BES linear system (3.3) introducing the even/odd Neumann expansion⁸

$$\sigma_+^{(0)}(\sqrt{2}gt) = \sum_{p=1}^{\infty} S_{2p}^{(0)}(g) J_{2p}(\sqrt{2}gt), \quad \sigma_-^{(0)}(\sqrt{2}gt) = \sum_{p=1}^{\infty} S_{2p-1}^{(0)}(g) J_{2p-1}(\sqrt{2}gt), \quad (4.28)$$

with the coefficients $S_r^{(0)}(g)$ given by

$$S_{2p}^{(0)}(g) = 2(2p) \int_0^{+\infty} \frac{dt}{t} \sigma_+^{(0)}(t) J_{2p}(t), \quad S_{2p-1}^{(0)}(g) = 2(2p-1) \int_0^{+\infty} \frac{dt}{t} \sigma_-^{(0)}(t) J_{2p-1}(t). \quad (4.29)$$

Then, the BES linear system can be cast in the form [12]

$$\begin{aligned} \int_0^{+\infty} \frac{dt}{t} \left[\frac{\sigma_+^{(0)}(\sqrt{2}gt)}{1-e^{-t}} - \frac{\sigma_-^{(0)}(\sqrt{2}gt)}{e^t-1} \right] J_{2p}(\sqrt{2}gt) &= 0, \\ \int_0^{+\infty} \frac{dt}{t} \left[\frac{\sigma_-^{(0)}(\sqrt{2}gt)}{1-e^{-t}} + \frac{\sigma_+^{(0)}(\sqrt{2}gt)}{e^t-1} \right] J_{2p-1}(\sqrt{2}gt) &= \sqrt{2}g \delta_{1,p}. \end{aligned} \quad (4.30)$$

Since the kernel of the reduced system (4.23) is the same as the BES one (3.3), it is possible to use the same procedure introducing the functions

$$\sigma_+^{(k)}(\sqrt{2}gt) = \sum_{p=1}^{+\infty} \tilde{S}_{2p}^{(k)}(g) J_{2p}(\sqrt{2}gt), \quad \sigma_-^{(k)}(\sqrt{2}gt) = \sum_{p=1}^{+\infty} \tilde{S}_{2p-1}^{(k)}(g) J_{2p-1}(\sqrt{2}gt), \quad (4.31)$$

together with

$$\tilde{S}_{2p}^{(k)}(g) = 2(2p) \int_0^{\infty} \frac{dt}{t} \sigma_+^{(k)}(t) J_{2p}(t), \quad \tilde{S}_{2p-1}^{(k)}(g) = 2(2p-1) \int_0^{\infty} \frac{dt}{t} \sigma_-^{(k)}(t) J_{2p-1}(t). \quad (4.32)$$

And, from the system (4.23), we derive the following equations for the functions $\sigma_{\pm}^{(k)}(t)$:

$$\begin{aligned} \int_0^{+\infty} \frac{dt}{t} \left[\frac{\sigma_+^{(k)}(\sqrt{2}gt)}{1-e^{-t}} - \frac{\sigma_-^{(k)}(\sqrt{2}gt)}{e^t-1} \right] J_{2p}(\sqrt{2}gt) &= \frac{1}{4\pi} \int_0^{+\infty} dt \frac{t^{2k-1}}{\sinh t/2} J_{2p}(\sqrt{2}gt), \\ \int_0^{+\infty} \frac{dt}{t} \left[\frac{\sigma_-^{(k)}(\sqrt{2}gt)}{1-e^{-t}} + \frac{\sigma_+^{(k)}(\sqrt{2}gt)}{e^t-1} \right] J_{2p-1}(\sqrt{2}gt) &= \frac{1}{4\pi} \int_0^{+\infty} dt \frac{t^{2k-1}}{\sinh t/2} J_{2p-1}(\sqrt{2}gt). \end{aligned} \quad (4.33)$$

The next step is to perform some manipulations on systems (4.30, 4.33), in order to exploit their similarities. Concentrating first on (4.30), we multiply both sides of the first equation by $\tilde{S}_{2p}^{(k)}(g)$,

⁸The use of $\sigma_{\pm}^{(0)}(\sqrt{2}gt)$ is redundant with respect to $S^{(0)}(k)$ (3.2). However, since in Appendix A we will use results of [16], we prefer to use here notations of [16].

and both sides of the second equation by $\tilde{S}_{2p-1}^{(k)}(g)$. Summing over p in both of them, we end up with

$$\begin{aligned} \int_0^{+\infty} \frac{dt}{t} \left[\frac{\sigma_+^{(0)}(\sqrt{2}gt) \sigma_+^{(k)}(\sqrt{2}gt)}{1 - e^{-t}} - \frac{\sigma_-^{(0)}(\sqrt{2}gt) \sigma_+^{(k)}(\sqrt{2}gt)}{e^t - 1} \right] &= 0, \\ \int_0^{+\infty} \frac{dt}{t} \left[\frac{\sigma_-^{(0)}(\sqrt{2}gt) \sigma_-^{(k)}(\sqrt{2}gt)}{1 - e^{-t}} + \frac{\sigma_+^{(0)}(\sqrt{2}gt) \sigma_-^{(k)}(\sqrt{2}gt)}{e^t - 1} \right] &= \sqrt{2}g \tilde{S}_1^{(k)}(g), \end{aligned}$$

where we notice that the coefficient $\tilde{S}_1^{(k)}(g)$ is explicitly singled out.

The same procedure can be repeated upon (4.33), by multiplying the first equation by $S_{2p}^{(0)}(g)$, the second by $S_{2p-1}^{(0)}(g)$ and finally summing over p . The result is as follows:

$$\begin{aligned} \int_0^{+\infty} \frac{dt}{t} \left[\frac{\sigma_+^{(0)}(\sqrt{2}gt) \sigma_+^{(k)}(\sqrt{2}gt)}{1 - e^{-t}} - \frac{\sigma_+^{(0)}(\sqrt{2}gt) \sigma_-^{(k)}(\sqrt{2}gt)}{e^t - 1} \right] &= \frac{1}{4\pi} \int_0^{+\infty} dt \frac{t^{2k-1}}{\sinh t/2} \sigma_+^{(0)}(\sqrt{2}gt), \\ \int_0^{+\infty} \frac{dt}{t} \left[\frac{\sigma_-^{(0)}(\sqrt{2}gt) \sigma_-^{(k)}(\sqrt{2}gt)}{1 - e^{-t}} + \frac{\sigma_-^{(0)}(\sqrt{2}gt) \sigma_+^{(k)}(\sqrt{2}gt)}{e^t - 1} \right] &= \frac{1}{4\pi} \int_0^{+\infty} dt \frac{t^{2k-1}}{\sinh t/2} \sigma_-^{(0)}(\sqrt{2}gt). \end{aligned}$$

A direct comparison with the previous equations allows to eventually obtain the integral representation for $\tilde{S}_1^{(k)}(g)$,

$$\sqrt{2}g \tilde{S}_1^{(k)}(g) = -\frac{1}{4\pi} \int_0^{+\infty} dt \frac{t^{2k-1}}{\sinh t/2} \left[\sigma_+^{(0)}(\sqrt{2}gt) - \sigma_-^{(0)}(\sqrt{2}gt) \right]. \quad (4.34)$$

For what concerns the coefficient $S_1^1(g)$, relevant for the computation of $f_1(g)$, the procedure is identical - one starts from (3.9) - but the result is slightly different. We have

$$\sqrt{2}g S_1^1(g) = - \int_0^\infty \frac{dt}{t} \frac{1}{2 \cosh t/4} \left[e^{-\frac{t}{4}} \sigma_-^{(0)}(\sqrt{2}gt) + e^{\frac{t}{4}} \sigma_+^{(0)}(\sqrt{2}gt) \right]. \quad (4.35)$$

Equations (4.34, 4.35) are the representations of, respectively, $\tilde{S}_1^{(k)}(g)$ and $S_1^1(g)$ in terms of the BES density.

Now, for completeness' sake and since we will be using it, we write also the integral representation in terms of the solution of the BES equation for the density and its derivatives in zero $\sigma^{(0);(s)}$. Such representation trivially follows from definition (3.1):

$$\begin{aligned} i^{-s} d_s \sigma^{(0);(s)} &= -4\delta_{s,0} + i^{-s} d_s \sigma_H^{(0);(s)}, \\ i^{-s} d_s \sigma_H^{(0);(s)} &= \int_0^{+\infty} dk \frac{k^s}{\sinh k/2} \sum_{p=1}^{\infty} S_p^{(0)}(g) J_p(\sqrt{2}gk) = \\ &= \int_0^{+\infty} dk \frac{k^s}{\sinh k/2} \left[\sigma_+^{(0)}(\sqrt{2}gk) + \sigma_-^{(0)}(\sqrt{2}gk) \right]. \end{aligned} \quad (4.36)$$

We are naturally arrived at one of the main points of our work and some comments are now in order. We have found that the generalised scaling functions $f_n(g)$, $n \geq 2$ enjoy an expression (4.22)

in terms of the 'reduced' coefficients $\tilde{S}_1^{(k)}(g)$ - related to the solution of the BES equation - and of the densities and their derivatives at zero $\sigma^{(n');(s)}$, $n' \leq n-3$ - related (when $n' \geq 1$) also to the other, 'higher', systems of equations (3.9), (4.23).

From the physical point of view the quantities $\tilde{S}_1^{(k)}(g)$ and $\sigma^{(n');(s)}$ are expressed in terms of the various masses of the SYM theory. In next section, we will show that all these quantities can be explicitly computed in the strong coupling limit by using iterative methods. In such a limit they will be all expressed in terms of one mass, the mass gap of the $O(6)$ nonlinear sigma model, embedded in the $\mathcal{N} = 4$ SYM theory.

5 Explicit results at strong coupling

In this section we will compute the strong coupling limit of two sets of quantities: on the one hand $\sigma^{(0);(s)}$ and $\tilde{S}_1^{(k)}(g)$, on the other hand $\sigma^{(n');(s)}$, with $n' \geq 1$, and $c^{(m)}$.

The first set of quantities can be computed by relying on the solution of the BES problem. Indeed, a simple but long calculation, which was done using the method developed in [16] and whose details are given in Appendix A, allows to give the following analytic estimates at strong coupling,

$$\begin{aligned}\sqrt{2}g\tilde{S}_1^{(k)}(g) &= \frac{(-1)^{k+1}}{4\pi} \left(\frac{\pi}{2}\right)^{2k} 2m(g) + O(e^{-3\frac{g\pi}{\sqrt{2}}}), \\ \sigma_H^{(0);(2k)} &= -\left(\frac{\pi}{2}\right)^{2k} \pi m(g) + O(e^{-3\frac{g\pi}{\sqrt{2}}}), \quad k > 0, \\ \sigma_H^{(0);(0)} &= 4 - \pi m(g) + O(e^{-3\frac{g\pi}{\sqrt{2}}}),\end{aligned}\tag{5.1}$$

where $m(g)$ (3.13) agrees at the leading order with the mass gap of the $O(6)$ NLSM. Using the first of (4.36) we can write, for the all loops density:

$$\begin{aligned}\sqrt{2}g\tilde{S}_1^{(k)}(g) &= \frac{(-1)^{k+1}}{4\pi} \left(\frac{\pi}{2}\right)^{2k} 2m(g) + O(e^{-3\frac{g\pi}{\sqrt{2}}}), \\ \sigma^{(0);(2k)} &= -\left(\frac{\pi}{2}\right)^{2k} \pi m(g) + O(e^{-3\frac{g\pi}{\sqrt{2}}}).\end{aligned}\tag{5.2}$$

For what concerns the $\sigma^{(n');(s)}$, with $n' \geq 1$, and the $c^{(m)}$, we first solve at strong coupling the system (4.23) for the $\tilde{S}_r^{(k)}(g)$, finding an expression for such coefficients as asymptotic series in inverse powers of g . Then, using this expression, we will write a recursive equation relating the large g leading behaviour of the quantities $\sigma^{(n');(s)}$, with $n' \geq 2$, and the constants $c^{(m)}$. This equation - together with (4.27) - will allow to find recursively both $\sigma^{(n');(s)}$, with $n' \geq 2$, and the $c^{(m)}$.

Having exposed the plan of our work, let us start from the reduced system (4.23) and let us look for a solution of it in the form

$$\tilde{S}_{2m}^{(k)}(g) = \sum_{n=k}^{\infty} \frac{\tilde{S}_{2m}^{(k;2n)}}{g^{2n}}, \quad \tilde{S}_{2m-1}^{(k)}(g) = \sum_{n=k+1}^{\infty} \frac{\tilde{S}_{2m-1}^{(k;2n-1)}}{g^{2n-1}},\tag{5.3}$$

with

$$\tilde{S}_{2m}^{(k;2n)} = 2m \frac{\Gamma(m+n)}{\Gamma(m-n+1)} (-1)^{1+n} \tilde{b}_{2n}^{(k)}, \quad n \geq k, \quad (5.4)$$

$$\tilde{S}_{2m-1}^{(k;2n-1)} = (2m-1) \frac{\Gamma(m+n-1)}{\Gamma(m-n+1)} (-1)^n \tilde{b}_{2n-1}^{(k)}, \quad n \geq k+1. \quad (5.5)$$

Usual techniques [15, 17] allow to find the unknown $\tilde{b}_{2n-1}^{(k)}, \tilde{b}_{2n}^{(k)}$ as solutions of the two recursive equations

$$\tilde{b}_{2n}^{(k)} = \sum_{m=0}^{n-k} (-1)^m 2^{m+\frac{1}{2}} \frac{\tilde{b}_{2n-2m+1}^{(k)}}{(2m)!} B_{2m}, \quad n \geq k, \quad (5.6)$$

$$\tilde{b}_{2n+1}^{(k)} = \frac{(-1)^n}{2\pi} 2^{2k+\frac{1}{2}-n} \frac{2^{2n-2k+1} - 1}{(2n-2k+2)!} B_{2n-2k+2} + \sum_{m=0}^{n+1-k} (-1)^m 2^{m+\frac{1}{2}} \frac{\tilde{b}_{2n+2-2m}^{(k)}}{(2m)!} B_{2m}, \quad n \geq k.$$

By comparing such equations with the corresponding equations for the coefficients b_N appearing in the asymptotic solution for $S_N^{(1)}(g)$ we find the simple correspondence

$$\tilde{b}_N^{(k)} = \frac{(-1)^{k+1} 2^k}{2\pi} b_{N-2k}, \quad N \geq 2k. \quad (5.7)$$

Putting all the relevant relations inside (4.22) and redefining (for conciseness' sake) the indexes l and s , we finally find the asymptotic expansions⁹

$$\begin{aligned} S_{2p}^{(n)}(g) &= \frac{2p}{\pi} \sum_{k=1}^n \sum_{l=1}^{\lfloor \frac{k-1}{2} \rfloor} (-1)^l \left[\sum_{s=0}^{l-1} \binom{2l}{2s} (-1)^s \sigma^{(n-k);(2s)} \right. \\ &\quad \cdot \left. \sum_{n'=0}^{\infty} \frac{2^{l-s} (-1)^{n'}}{g^{2n'+2l-2s}} \frac{\Gamma(p+n'+l-s)}{\Gamma(p-n'-l+s+1)} b_{2n'} \right] \sum_{\{j_1, \dots, j_{k-2l}\}} \prod_{m=1}^{k-2l} \frac{(c^{(m)})^{j_m}}{j_m!}, \end{aligned} \quad (5.8)$$

and

$$\begin{aligned} S_{2p-1}^{(n)}(g) &= \frac{2p-1}{\pi} \sum_{k=1}^n \sum_{l=1}^{\lfloor \frac{k-1}{2} \rfloor} (-1)^l \left[\sum_{s=0}^{l-1} \binom{2l}{2s} (-1)^s \sigma^{(n-k);(2s)} \right. \\ &\quad \cdot \left. \sum_{n'=0}^{\infty} \frac{2^{l-s} (-1)^{n'}}{g^{2n'+2l-2s+1}} \frac{\Gamma(p+n'+l-s)}{\Gamma(p-n'-l+s)} b_{2n'+1} \right] \sum_{\{j_1, \dots, j_{k-2l}\}} \prod_{m=1}^{k-2l} \frac{(c^{(m)})^{j_m}}{j_m!}, \end{aligned} \quad (5.9)$$

where the positive integers j_m have to satisfy

$$\sum_{m=1}^{k-2l} j_m = 2l+1, \quad \sum_{m=1}^{k-2l} m j_m = k. \quad (5.10)$$

⁹The notation $[x]$ present in (5.8, 5.9) stands for the integer part of the semi-integer x .

We are now ready to discuss the strong coupling behaviour of the densities of Bethe roots and their derivatives at $u = 0$, $\sigma^{(n);(s)}$, when $n \geq 1$. We have to distinguish between the case $n = 1$ and the cases $n \geq 2$.

In the case $n = 1$ we have that

$$\sigma_0^{(1);(0)} = -4 \ln 2, \quad i^{-s} d_s \sigma_0^{(1);(s)} = (4 - 2^{s+2}) \Gamma(s+1) \zeta(s+1), \quad s \geq 2, \quad (5.11)$$

for the one loop theory and

$$i^{-s} d_s \sigma_H^{(1);(s)} = \int_0^{+\infty} dk \frac{k^s}{\sinh k/2} \sum_{p=1}^{\infty} S_p^{(1)}(g) J_p(\sqrt{2}gk), \quad (5.12)$$

for the higher than one loop contributions. We now insert in (5.12) the asymptotic solution [15] for $S_p^{(1)}(g)$, reported also in (3.10, 3.11). Summing first over p and then over n' and finally integrating in k we get

$$\begin{aligned} \sigma_H^{(1);(0)} &= 3 \ln 2 - \frac{\pi}{2}, \\ i^{-s} d_s \sigma_H^{(1);(s)} &= (2^{s+2} - 4 + 2^{-2s} - 2^{-s}) \Gamma(s+1) \zeta(s+1) - \left(\frac{\pi}{2}\right)^{s+1} |E_s|, \quad s \geq 2, \end{aligned} \quad (5.13)$$

where E_k are the Euler's numbers. Summing with the one loop results, one gets the explicit formula

$$\begin{aligned} \sigma^{(1);(0)} &= -\ln 2 - \frac{\pi}{2}, \\ i^{-s} d_s \sigma^{(1);(s)} &= (2^{-2s} - 2^{-s}) \Gamma(s+1) \zeta(s+1) - \left(\frac{\pi}{2}\right)^{s+1} |E_s|, \quad s \geq 2. \end{aligned} \quad (5.14)$$

For the case $n \geq 2$ we start from (3.14) and write

$$i^{-s} d_s \sigma^{(n);(s)} = 2 \int_0^{+\infty} \frac{dk}{2\pi} k^s \left[\frac{\pi k}{\sinh \frac{k}{2}} S^{(n)}(k) - \frac{e^{-\frac{k}{2}}}{\sinh \frac{k}{2}} \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \hat{\sigma}(p) \frac{\sin(k-p)c}{k-p} \Big|_{j^n} \right]. \quad (5.15)$$

We then insert (5.8, 5.9) in the expansion (3.17) of $S^{(n)}(k)$ in series of Bessel functions and use (4.20) to express the integral term in the square brackets. Summing on the indexes p and n' coming from (5.8, 5.9), we end up with the previously announced recursive equation,

$$\begin{aligned} i^{-s} d_s \sigma^{(n);(s)} &= \sum_{k=1}^n \sum_{l=1}^{\lfloor \frac{k-1}{2} \rfloor} (-1)^l \sum_{s'=0}^{l-1} \binom{2l}{2s'} (-1)^{s'} \sigma^{(n-k);(2s')} . \\ &\cdot \int_0^{+\infty} \frac{dk}{\pi} \frac{k^{s+2l-2s'}}{\sinh \frac{k}{2}} \left(\frac{e^{\frac{k}{2}}}{\cosh k} - e^{-\frac{k}{2}} \right) \sum_{\{j_1, \dots, j_{k-2l}\}} \prod_{m=1}^{k-2l} \frac{(c^{(m)})^{j_m}}{j_m!} = \\ &= \sum_{k=1}^n \sum_{l=1}^{\lfloor \frac{k-1}{2} \rfloor} (-1)^l \sum_{s'=0}^{l-1} \binom{2l}{2s'} (-1)^{s'} \sigma^{(n-k);(2s')} . \\ &\cdot \frac{1}{\pi} \left[(2^{-s-2l+2s'} - 2^{-2s-4l+4s'}) \Gamma(s+2l-2s'+1) \zeta(s+2l-2s'+1) + \right. \\ &\left. + \left(\frac{\pi}{2}\right)^{s+2l-2s'+1} |E_{s+2l-2s'}| \right] \sum_{\{j_1, \dots, j_{k-2l}\}} \prod_{m=1}^{k-2l} \frac{(c^{(m)})^{j_m}}{j_m!}, \end{aligned} \quad (5.16)$$

where, again,

$$\sum_{m=1}^{k-2l} j_m = 2l + 1, \quad \sum_{m=1}^{k-2l} m j_m = k. \quad (5.17)$$

As we said before, such equation has to be solved together with (4.27).

To summarize the results, in this section we have shown that, similarly to the case of $S_r^{(1)}(g)$ [15], also the system for the $\tilde{S}_r^{(k)}(g)$ can be solved at large g . This allowed to write for the $\sigma^{(n');(s)}$ a further recursion relation (5.16) - which goes together with the explicit expressions (5.14, 5.2), coming from the solution of the systems for $S_r^{(1)}(g)$ and from results of [12], respectively. In this regime, the set of constants $c^{(m)}$ and the set of densities $\sigma^{(n');(k)}$ can be computed by solving simultaneously the recursive relations (4.27) and (5.16). Putting their expressions, together with the solution for $\tilde{S}_1^{(k)}(g)$, into (4.22), one will get the expression for $S_1^{(n)}(g)$ and, consequently, for $f_n(g) = \sqrt{2}gS_1^{(n)}(g)$ at strong coupling.

As an application of all these techniques, in the next section we will compute explicitly the strong coupling limit of the scaling functions $f_3(g), \dots, f_8(g)$.

5.1 Examples: $f_3(g)$ to $f_8(g)$

The previous machinery can be tested by computing the strong coupling behaviour of $f_n(g)$, for $3 \leq n \leq 8$, in order to compare it with the available result from the $O(6)$ NLSM.

First of all, we need to know the expression of $c^{(1)}, \dots, c^{(6)}$. From the recursion formula (4.27) we get

$$c^{(1)} = -\frac{\pi}{\sigma^{(0);(0)}}, \quad (5.18)$$

$$c^{(2)} = \pi \frac{\sigma^{(1);(0)}}{[\sigma^{(0);(0)}]^2}, \quad (5.19)$$

$$c^{(3)} = \frac{\pi^3}{6} \frac{\sigma^{(0);(2)}}{[\sigma^{(0);(0)}]^4} - \pi \frac{[\sigma^{(1);(0)}]^2}{[\sigma^{(0);(0)}]^3}, \quad (5.20)$$

$$c^{(4)} = \pi \frac{\sigma^{(3);(0)}}{[\sigma^{(0);(0)}]^2} - \frac{2}{3} \pi^3 \frac{\sigma^{(0);(2)} \sigma^{(1);(0)}}{[\sigma^{(0);(0)}]^5} + \pi \frac{[\sigma^{(1);(0)}]^3}{[\sigma^{(0);(0)}]^4} + \frac{1}{6} \pi^3 \frac{\sigma^{(1);(2)}}{[\sigma^{(0);(0)}]^4}, \quad (5.21)$$

$$\begin{aligned} c^{(5)} = & -\frac{\pi [\sigma^{(1);(0)}]^4}{[\sigma^{(0);(0)}]^5} + \frac{5\pi^3 \sigma^{(0);(2)} [\sigma^{(1);(0)}]^2}{3[\sigma^{(0);(0)}]^6} - \frac{2\pi^3 \sigma^{(1);(2)} \sigma^{(1);(0)}}{3[\sigma^{(0);(0)}]^5} - \frac{2\pi \sigma^{(3);(0)} \sigma^{(1);(0)}}{[\sigma^{(0);(0)}]^3} + \\ & - \frac{\pi^5 [\sigma^{(0);(2)}]^2}{12[\sigma^{(0);(0)}]^7} + \frac{\pi^5 \sigma^{(0);(4)}}{120[\sigma^{(0);(0)}]^6} + \frac{\pi \sigma^{(4);(0)}}{[\sigma^{(0);(0)}]^2}, \end{aligned} \quad (5.22)$$

$$\begin{aligned} c^{(6)} = & \frac{\pi [\sigma^{(1);(0)}]^5}{[\sigma^{(0);(0)}]^6} - \frac{10\pi^3 \sigma^{(0);(2)} [\sigma^{(1);(0)}]^3}{3[\sigma^{(0);(0)}]^7} + \frac{5\pi^3 \sigma^{(1);(2)} [\sigma^{(1);(0)}]^2}{3[\sigma^{(0);(0)}]^6} + \frac{3\pi \sigma^{(3);(0)} [\sigma^{(1);(0)}]^2}{[\sigma^{(0);(0)}]^4} + \\ & + \frac{7\pi^5 [\sigma^{(0);(2)}]^2 \sigma^{(1);(0)}}{12[\sigma^{(0);(0)}]^8} - \frac{\pi^5 \sigma^{(0);(4)} \sigma^{(1);(0)}}{20[\sigma^{(0);(0)}]^7} - \frac{2\pi \sigma^{(4);(0)} \sigma^{(1);(0)}}{[\sigma^{(0);(0)}]^3} - \frac{\pi^5 \sigma^{(0);(2)} \sigma^{(1);(2)}}{6[\sigma^{(0);(0)}]^7} + \\ & + \frac{\pi^5 [\sigma^{(1);(4)}]}{120[\sigma^{(0);(0)}]^6} - \frac{2\pi^3 \sigma^{(0);(2)} \sigma^{(3);(0)}}{3[\sigma^{(0);(0)}]^5} + \frac{\pi^3 [\sigma^{(3);(2)}]}{6[\sigma^{(0);(0)}]^4} + \frac{\pi [\sigma^{(5);(0)}]}{[\sigma^{(0);(0)}]^2}. \end{aligned} \quad (5.23)$$

Then, referring for the notations to (4.20), we have

$$2 \int_{-\infty}^{\infty} \frac{dp}{2\pi} [\Gamma_3(h-p, p) - \Gamma_3(p, p)] = \frac{1}{3} \pi^3 \frac{h^2}{[\sigma^{(0);(0)}]^2} \quad (5.24)$$

$$2 \int_{-\infty}^{\infty} \frac{dp}{2\pi} [\Gamma_4(h-p, p) - \Gamma_4(p, p)] = -\frac{2}{3} \pi^3 \frac{h^2 \sigma^{(1);(0)}}{[\sigma^{(0);(0)}]^3} \quad (5.25)$$

$$2 \int_{-\infty}^{\infty} \frac{dp}{2\pi} [\Gamma_5(h-p, p) - \Gamma_5(p, p)] = -\frac{\pi^5 h^4}{60 [\sigma^{(0);(0)}]^4} + \frac{\pi^3 [\sigma^{(1);(0)}]^2 h^2}{[\sigma^{(0);(0)}]^4} - \frac{\pi^5 \sigma^{(0);(2)} h^2}{15 [\sigma^{(0);(0)}]^5} \quad (5.26)$$

$$2 \int_{-\infty}^{\infty} \frac{dp}{2\pi} [\Gamma_6(h-p, p) - \Gamma_6(p, p)] = \frac{\pi^5 h^4 \sigma^{(1);(0)}}{15 [\sigma^{(0);(0)}]^5} - \frac{2}{3} \frac{\pi^3 \sigma^{(3);(0)} h^2}{[\sigma^{(0);(0)}]^3} + \frac{1}{3} \frac{\pi^5 \sigma^{(1);(0)} \sigma^{(0);(2)} h^2}{[\sigma^{(0);(0)}]^6} +$$

$$- \frac{4}{3} \frac{\pi^3 [\sigma^{(1);(0)}]^3 h^2}{[\sigma^{(0);(0)}]^5} - \frac{1}{15} \frac{\pi^5 \sigma^{(1);(2)} h^2}{[\sigma^{(0);(0)}]^5} \quad (5.27)$$

$$2 \int_{-\infty}^{\infty} \frac{dp}{2\pi} [\Gamma_7(h-p, p) - \Gamma_7(p, p)] = \frac{\pi^7 h^6}{2520 [\sigma^{(0);(0)}]^6} - \frac{\pi^5 [\sigma^{(1);(0)}]^2 h^4}{6 [\sigma^{(0);(0)}]^6} + \frac{\pi^7 \sigma^{(0);(2)} h^4}{126 [\sigma^{(0);(0)}]^7} +$$

$$+ \frac{5\pi^3 [\sigma^{(1);(0)}]^4 h^2}{3 [\sigma^{(0);(0)}]^6} + \frac{\pi^7 [\sigma^{(0);(2)}]^2 h^2}{36 [\sigma^{(0);(0)}]^8} - \frac{\pi^5 \sigma^{(0);(2)} [\sigma^{(1);(0)}]^2 h^2}{[\sigma^{(0);(0)}]^7} - \frac{\pi^7 \sigma^{(0);(4)} h^2}{420 [\sigma^{(0);(0)}]^7} + \frac{\pi^5 [\sigma^{(1);(0)}] \sigma^{(1);(2)} h^2}{3 [\sigma^{(0);(0)}]^6} +$$

$$+ \frac{2\pi^3 [\sigma^{(1);(0)}] \sigma^{(3);(0)} h^2}{[\sigma^{(0);(0)}]^4} - \frac{2\pi^3 \sigma^{(4);(0)} h^2}{3 [\sigma^{(0);(0)}]^3} \quad (5.28)$$

$$2 \int_{-\infty}^{\infty} \frac{dp}{2\pi} [\Gamma_8(h-p, p) - \Gamma_8(p, p)] = -\frac{\pi^7 [\sigma^{(1);(0)}] h^6}{420 [\sigma^{(0);(0)}]^7} + \frac{\pi^5 [\sigma^{(1);(0)}]^3 h^4}{3 [\sigma^{(0);(0)}]^7} - \frac{\pi^7 \sigma^{(0);(2)} [\sigma^{(1);(0)}] h^4}{18 [\sigma^{(0);(0)}]^8} +$$

$$+ \frac{\pi^7 \sigma^{(1);(2)} h^4}{126 [\sigma^{(0);(0)}]^7} + \frac{\pi^5 \sigma^{(3);(0)} h^4}{15 [\sigma^{(0);(0)}]^5} - \frac{2\pi^3 [\sigma^{(1);(0)}]^5 h^2}{[\sigma^{(0);(0)}]^7} + \frac{7\pi^5 \sigma^{(0);(2)} [\sigma^{(1);(0)}]^3 h^2}{3 [\sigma^{(0);(0)}]^8} - \frac{2\pi^7 [\sigma^{(0);(2)}]^2 [\sigma^{(1);(0)}] h^2}{9 [\sigma^{(0);(0)}]^9} +$$

$$+ \frac{\pi^7 \sigma^{(0);(4)} [\sigma^{(1);(0)}] h^2}{60 [\sigma^{(0);(0)}]^8} - \frac{\pi^5 [\sigma^{(1);(0)}]^2 \sigma^{(1);(2)} h^2}{[\sigma^{(0);(0)}]^7} + \frac{\pi^7 \sigma^{(0);(2)} \sigma^{(1);(2)} h^2}{18 [\sigma^{(0);(0)}]^8} - \frac{\pi^7 [\sigma^{(1);(4)}] h^2}{420 [\sigma^{(0);(0)}]^7} -$$

$$- \frac{4\pi^3 [\sigma^{(1);(0)}]^2 \sigma^{(3);(0)} h^2}{[\sigma^{(0);(0)}]^5} + \frac{\pi^5 \sigma^{(0);(2)} \sigma^{(3);(0)} h^2}{3 [\sigma^{(0);(0)}]^6} - \frac{\pi^5 [\sigma^{(3);(2)}] h^2}{15 [\sigma^{(0);(0)}]^5} + \frac{2\pi^3 [\sigma^{(1);(0)}] \sigma^{(4);(0)} h^2}{[\sigma^{(0);(0)}]^4} -$$

$$- \frac{2\pi^3 [\sigma^{(5);(0)}] h^2}{3 [\sigma^{(0);(0)}]^3}. \quad (5.29)$$

This implies that for $f_3(g), \dots, f_8(g)$ we can give the exact (i.e. valid $\forall g$) expressions:

$$\frac{f_3(g)}{2\sqrt{2}g} = \frac{1}{6}\pi^3 \frac{1}{[\sigma^{(0);(0)}]^2} \tilde{S}_1^{(1)}(g), \quad (5.30)$$

$$\frac{f_4(g)}{2\sqrt{2}g} = -\frac{1}{3}\pi^3 \frac{\sigma^{(1);(0)}}{[\sigma^{(0);(0)}]^3} \tilde{S}_1^{(1)}(g), \quad (5.31)$$

$$\frac{f_5(g)}{2\sqrt{2}g} = -\frac{\pi^5}{120[\sigma^{(0);(0)}]^4} \tilde{S}_1^{(2)}(g) + \frac{1}{2} \frac{\pi^3 [\sigma^{(1);(0)}]^2}{[\sigma^{(0);(0)}]^4} \tilde{S}_1^{(1)}(g) - \frac{\pi^5 \sigma^{(0);(2)}}{30[\sigma^{(0);(0)}]^5} \tilde{S}_1^{(1)}(g), \quad (5.32)$$

$$\begin{aligned} \frac{f_6(g)}{2\sqrt{2}g} &= \frac{\pi^5 \sigma^{(1);(0)}}{30[\sigma^{(0);(0)}]^5} \tilde{S}_1^{(2)}(g) + \left[-\frac{1}{3} \frac{\pi^3 \sigma^{(3);(0)}}{[\sigma^{(0);(0)}]^3} + \right. \\ &+ \left. \frac{1}{6} \frac{\pi^5 \sigma^{(1);(0)} \sigma^{(0);(2)}}{[\sigma^{(0);(0)}]^6} - \frac{2}{3} \frac{\pi^3 [\sigma^{(1);(0)}]^3}{[\sigma^{(0);(0)}]^5} - \frac{1}{30} \frac{\pi^5 \sigma^{(1);(2)}}{[\sigma^{(0);(0)}]^5} \right] \tilde{S}_1^{(1)}(g), \end{aligned} \quad (5.33)$$

$$\begin{aligned} \frac{f_7(g)}{2\sqrt{2}g} &= \frac{\pi^7}{5040[\sigma^{(0);(0)}]^6} \tilde{S}_1^{(3)}(g) + \left[-\frac{\pi^5 [\sigma^{(1);(0)}]^2}{12[\sigma^{(0);(0)}]^6} + \frac{\pi^7 \sigma^{(0);(2)}}{252[\sigma^{(0);(0)}]^7} \right] \tilde{S}_1^{(2)}(g) + \\ &+ \left[\frac{5\pi^3 [\sigma^{(1);(0)}]^4}{6[\sigma^{(0);(0)}]^6} + \frac{\pi^7 [\sigma^{(0);(2)}]^2}{72[\sigma^{(0);(0)}]^8} - \frac{\pi^5 \sigma^{(0);(2)} [\sigma^{(1);(0)}]^2}{2[\sigma^{(0);(0)}]^7} - \frac{\pi^7 \sigma^{(0);(4)}}{840[\sigma^{(0);(0)}]^7} + \right. \\ &+ \left. \frac{\pi^5 [\sigma^{(1);(0)}] \sigma^{(1);(2)}}{6[\sigma^{(0);(0)}]^6} + \frac{\pi^3 [\sigma^{(1);(0)}] \sigma^{(3);(0)}}{[\sigma^{(0);(0)}]^4} - \frac{\pi^3 \sigma^{(4);(0)}}{3[\sigma^{(0);(0)}]^3} \right] \tilde{S}_1^{(1)}(g), \end{aligned} \quad (5.34)$$

$$\begin{aligned} \frac{f_8(g)}{2\sqrt{2}g} &= -\frac{\pi^7 [\sigma^{(1);(0)}]}{840[\sigma^{(0);(0)}]^7} \tilde{S}_1^{(3)}(g) + \left[\frac{\pi^5 [\sigma^{(1);(0)}]^3}{6[\sigma^{(0);(0)}]^7} - \frac{\pi^7 \sigma^{(0);(2)} [\sigma^{(1);(0)}]}{36[\sigma^{(0);(0)}]^8} + \frac{\pi^7 \sigma^{(1);(2)}}{252[\sigma^{(0);(0)}]^7} + \right. \\ &+ \left. \frac{\pi^5 \sigma^{(3);(0)}}{30[\sigma^{(0);(0)}]^5} \right] \tilde{S}_1^{(2)}(g) + \left[-\frac{\pi^3 [\sigma^{(1);(0)}]^5}{[\sigma^{(0);(0)}]^7} + \frac{7\pi^5 \sigma^{(0);(2)} [\sigma^{(1);(0)}]^3}{6[\sigma^{(0);(0)}]^8} - \frac{\pi^7 [\sigma^{(0);(2)}]^2 [\sigma^{(1);(0)}]}{9[\sigma^{(0);(0)}]^9} + \right. \\ &+ \frac{\pi^7 \sigma^{(0);(4)} [\sigma^{(1);(0)}]}{120[\sigma^{(0);(0)}]^8} - \frac{\pi^5 [\sigma^{(1);(0)}]^2 \sigma^{(1);(2)}}{2[\sigma^{(0);(0)}]^7} + \frac{\pi^7 \sigma^{(0);(2)} \sigma^{(1);(2)}}{36[\sigma^{(0);(0)}]^8} - \frac{2\pi^3 [\sigma^{(1);(0)}]^2 \sigma^{(3);(0)}}{[\sigma^{(0);(0)}]^5} \\ &+ \left. \frac{\pi^5 \sigma^{(0);(2)} \sigma^{(3);(0)}}{6[\sigma^{(0);(0)}]^6} - \frac{\pi^5 [\sigma^{(3);(2)}]}{30[\sigma^{(0);(0)}]^5} + \frac{\pi^3 [\sigma^{(1);(0)}] \sigma^{(4);(0)}}{[\sigma^{(0);(0)}]^4} - \frac{\pi^7 [\sigma^{(1);(4)}]}{840[\sigma^{(0);(0)}]^7} - \frac{\pi^3 [\sigma^{(5);(0)}]}{3[\sigma^{(0);(0)}]^3} \right] \tilde{S}_1^{(1)}(g), \end{aligned} \quad (5.35)$$

As stated before, there formulæ express the $f_n(g)$'s in terms of the masses of the $\mathcal{N} = 4$ SYM theory.

In the limit $g \rightarrow \infty$, the quantity $\sigma^{(n);(s)}$ for $n \geq 1$ can be computed using (5.14) and solving together the recursive equations (5.16) and (4.27). In particular we have, for their leading contributions,

$$\sigma^{(1);(0)} = -\ln 2 - \frac{\pi}{2}, \quad \sigma^{(1);(2)} = \frac{3\zeta(3) + \pi^3}{8}, \quad \sigma^{(3);(0)} = \frac{\pi^2}{6[\sigma^{(0);(0)}]^2} \sigma^{(1);(2)}, \quad (5.36)$$

$$\sigma^{(1);(4)} = -\frac{5}{32}(\pi^5 + 9\zeta(5)), \quad \sigma^{(3);(2)} = \frac{\pi^2}{6[\sigma^{(0);(0)}]^2} \sigma^{(1);(4)}, \quad \sigma^{(4);(0)} = -\pi^2 \frac{\sigma^{(1);(0)} \sigma^{(1);(2)}}{3[\sigma^{(0);(0)}]^3}, \quad (5.37)$$

$$\sigma^{(5);(0)} = \frac{\pi^2 \sigma^{(1);(2)} [\sigma^{(1);(0)}]^2}{2[\sigma^{(0);(0)}]^4} - \frac{\pi^4 \sigma^{(0);(2)} \sigma^{(1);(2)}}{30[\sigma^{(0);(0)}]^5} + \frac{\pi^4 [\sigma^{(1);(4)}]}{120[\sigma^{(0);(0)}]^4}. \quad (5.38)$$

These data, together with the limiting leading values at strong coupling of $\sigma^{(0);(s)}$ and $\tilde{S}_1^{(k)}(g)$ (5.2), allow to obtain for $f_3(g), \dots, f_8(g)$ the following leading values as $g \rightarrow \infty$,

$$f_3(g) = \frac{\pi^2}{24m(g)} \quad (5.39)$$

$$f_4(g) = -\frac{\pi^2}{12[m(g)]^2} \mathcal{S}_1 \quad (5.40)$$

$$f_5(g) = -\frac{\pi^4}{640[m(g)]^3} + \frac{\pi^2}{8[m(g)]^3} [\mathcal{S}_1]^2 \quad (5.41)$$

$$f_6(g) = \frac{\pi^4}{[m(g)]^4} \left(\frac{\mathcal{S}_3}{90} - \frac{[\mathcal{S}_1]^3}{6\pi^2} + \frac{\mathcal{S}_1}{120} \right) \quad (5.42)$$

$$f_7(g) = \frac{\pi^6}{7182[m(g)]^5} - \frac{5\pi^4[\mathcal{S}_1]^2}{192[m(g)]^5} + \frac{5\pi^2[\mathcal{S}_1]^4}{24[m(g)]^5} - \frac{\pi^4 \mathcal{S}_1 \mathcal{S}_3}{18[m(g)]^5} \quad (5.43)$$

$$f_8(g) = -\frac{\pi^6 \mathcal{S}_1}{840[m(g)]^6} + \frac{\pi^4[\mathcal{S}_1]^3}{16[m(g)]^6} - \frac{\pi^2[\mathcal{S}_1]^5}{4[m(g)]^6} - \frac{\pi^6 \mathcal{S}_3}{560[m(g)]^6} + \frac{\pi^4 \mathcal{S}_1^2 \mathcal{S}_3}{6[m(g)]^6} - \frac{\pi^6 \mathcal{S}_5}{280[m(g)]^6} \quad (5.44)$$

where we used the compact notations:

$$\mathcal{S}_{2s+1} = \frac{1}{\pi^{2s+1}} \sum_{n=0}^{\infty} (-1)^n \left[\frac{1}{(n + \frac{1}{2})^{2s+1}} + \frac{1}{(n+1)^{2s+1}} \right]. \quad (5.45)$$

For instance, we have

$$\mathcal{S}_1 = \frac{1}{\pi} \ln 2 + \frac{1}{2}, \quad \mathcal{S}_3 = \frac{1}{4\pi^3} [3\zeta(3) + \pi^3], \quad \mathcal{S}_5 = \frac{5}{48\pi^5} [9\zeta(5) + \pi^5]. \quad (5.46)$$

After a lengthy but straightforward calculation it is possible to show that such expressions agree (at the leading order) with the corresponding formulæ computed in the framework of the $O(6)$ NLSM, i.e. the coefficients $2^{n-1}\Omega_n$ given by the general formulae of [18].

5.2 Numerical evaluation of the next to leading correction to the mass

In order to check results presented in the previous sections we numerically estimated (via a best fit procedure) the leading non-analytic contribution at strong coupling for various quantities:

$$\alpha_1 = f_1(g) + 1, \quad \alpha_2 = -\frac{\sigma^{(0);(0)}}{\pi}, \quad \alpha_3 = -4\frac{\sigma^{(0);(2)}}{\pi^3}, \quad (5.47)$$

$$\alpha_4 = \frac{8}{\pi} [\sqrt{2}g\tilde{S}_1^{(1)}], \quad \alpha_5 = -\frac{16}{\pi^3} [\sqrt{2}g\tilde{S}_1^{(2)}]. \quad (5.48)$$

All of them turned out to be proportional to the leading $O(6)$ mass and the actual value was in perfect agreement with the computation by Alday and Maldacena. As a result, we were naturally led to conjecture the exact value for the leading exponential behaviour.

$\delta_{i,j}$	1	2	3	4	5
1	0	-0.818(2)	-5.72(3)	1.64(1)	16.3(2)
2	0.818(2)	0	-4.96(1)	2.46(2)	17.2(2)
3	5.72(3)	4.96(1)	0	7.35(1)	22.04(15)
4	-1.64(1)	-2.46(2)	-7.35(1)	0	14.7(2)
5	-16.3(2)	-17.2(2)	-22.04(15)	-14.7(2)	0

Table 1: Values of the differences $\delta_{i,j}$. The index i runs along the rows, and the index j runs along the columns.

Furthermore, we are in the position to give a quite precise estimate of the next to leading correction to the mass $m(g)$, which is known to be of the form

$$m(g) = k g^{1/4} (1 + \frac{k_1}{g} + \dots) e^{-\frac{\pi}{\sqrt{2}}g}, \quad k = \frac{2^{5/8} \pi^{1/4}}{\Gamma(5/4)}. \quad (5.49)$$

The analysis of the data at our disposal gives access to the quantity k_1 , whose best fit estimate is

$$k_1 = -0.0164 \pm 0.0005. \quad (5.50)$$

It is interesting to point out that such a correction has to be the same for all the functions α'_i s at leading order in the strong coupling limit. We checked that it is actually the case, giving a further proof of the reliability of our computations.

As matter of facts, all the corrections in $1/g$, the Brezin-Zinn Justin function [22], should be the same for the previous observables, because the computations of appendix A show that the first discrepancies should appear at order $O(e^{-\frac{3\pi}{\sqrt{2}}g})$.

Such a fact can be checked numerically, by performing a numerical analysis of the differences $\Delta_{ij} = \alpha_i - \alpha_j$, $i < j$. With a best fit procedure, we have been able to verify that all the Δ 's actually behaves as $[m(g)]^3/g$ but with different amplitudes, namely

$$\Delta_{i,j} = \delta_{i,j} \frac{[m(g)]^3}{g} + \dots \quad (5.51)$$

A consistency check that is fulfilled the numerical amplitudes $\delta_{i,j}$ is that the relations which are not linearly independent are actually compatible within the numerical precision. The amplitudes are collected in table 1.

6 Summary and outlook

In this paper we have discussed a systematic method which aims at the computation of all the generalised scaling functions $f_n(g)$ appearing in the $sl(2)$ sector of $\mathcal{N} = 4$ SYM in the scaling limit (1.4).

In our approach the generalised scaling functions are expressed, for all values of g , by (4.22). All the ingredients entering such expression can be found on the one hand from the solution of the BES equation and on the other hand from the solutions of the systems (3.9) for $S_r^{(1)}(g)$ and (4.23) for the reduced coefficients $\tilde{S}_r^{(k)}(g)$. At strong coupling the explicit solutions to the systems for the reduced coefficients has been explicitly worked out (5.8, 5.9) at the leading order. Together with previous analogous results concerning the BES equation [12] and $S_r^{(1)}(g)$ [15], this eventually allowed to compute the strong coupling behaviour of the first eight generalised scaling functions (5.39-5.44). Our results agree with the analogous findings [18] coming from the $O(6)$ nonlinear sigma model.

For what concerns future work, one has to say that in the $sl(2)$ sector of $\mathcal{N} = 4$ SYM other regimes - e.g. the large j regime in the context of limit (1.4) [23] - and, more generally, limits different from (1.4) are relevant for comparisons with pure string theory results. In this respect, the limit $s, L \rightarrow \infty$, $g \rightarrow \infty$, $l = L/(g \ln s)$ fixed - the so-called 'semiclassical scaling limit' - has been widely studied [24]. Application of our techniques to this case is a possible future direction of investigations.

Finally, one has to mention the new line of research related to the recently discovered duality between $\mathcal{N} = 6$ super Chern-Simons (SCS) theory with $U(N) \times U(N)$ gauge group at level k and superstring theory in the $AdS_4 \times CP^3$ background, when N is large and the 't Hooft coupling $\lambda = N/k$ is kept fixed [25]. Integrability on the gauge side [26, 27] and on the string side of the duality [28] was shown. Bethe ansatz-like equations were proposed [29] for the SCS theory and tested in various ways [30]. It could be surely of interest to apply the techniques discussed in this paper also to this new field of activity.

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A Non-analytic terms at strong coupling

The aim of this subsection is the analytic computation at large g of the leading contributions to eqs. (4.34) by means of the techniques developed in [16].

In particular, we will calculate the large g behaviour of the following integrals,

$$\begin{aligned} \mathcal{B}_{2k-1}(g) &= \int_0^{+\infty} dt \frac{t^{2k-1}}{\sinh t/2} \left[\sigma_+^{(0)}(\sqrt{2}gt) - \sigma_-^{(0)}(\sqrt{2}gt) \right], \\ \mathcal{C}_{2k}(g) &= \int_0^{+\infty} dt \frac{t^{2k}}{\sinh t/2} \left[\sigma_+^{(0)}(\sqrt{2}gt) + \sigma_-^{(0)}(\sqrt{2}gt) \right]. \end{aligned}$$

First of all, we make use of the BKK transformation [12, 16],

$$2\sigma_{\pm}^{(0)}(t) = \left(1 - \frac{1}{\cosh \frac{t}{\sqrt{2}g}}\right) \Sigma_{\pm}^{(0)}(t) \pm \tanh \frac{t}{\sqrt{2}g} \Sigma_{\mp}^{(0)}(t), \quad (\text{A.1})$$

in order to rewrite the integrals as

$$\begin{aligned} \mathcal{B}_{2k-1}(g) &= - \int_0^{+\infty} \frac{dt}{t} \left(\frac{t}{\sqrt{2}g}\right)^{2k} \left[\frac{\sinh \frac{t}{2\sqrt{2}g}}{\cosh \frac{t}{\sqrt{2}g}} (\Sigma_-^{(0)}(t) - \Sigma_+^{(0)}(t)) - \frac{\cosh \frac{t}{2\sqrt{2}g}}{\cosh \frac{t}{\sqrt{2}g}} (\Sigma_-^{(0)}(t) + \Sigma_+^{(0)}(t)) \right] \\ \mathcal{C}_{2k}(g) &= \int_0^{+\infty} \frac{dt}{t} \left(\frac{t}{\sqrt{2}g}\right)^{2k} \left[\frac{\cosh \frac{t}{2\sqrt{2}g}}{\cosh \frac{t}{\sqrt{2}g}} (\Sigma_-^{(0)}(t) - \Sigma_+^{(0)}(t)) + \frac{\sinh \frac{t}{2\sqrt{2}g}}{\cosh \frac{t}{\sqrt{2}g}} (\Sigma_-^{(0)}(t) + \Sigma_+^{(0)}(t)) \right]. \end{aligned}$$

The BES equation can be rewritten in terms of the functions $\Sigma_{\pm}^{(0)}$ [16], with $|u| < 1$

$$\begin{aligned} \int_0^{+\infty} dt \sin(ut) [\Sigma_-^{(0)}(t) + \Sigma_+^{(0)}(t)] &= 0, \\ \int_0^{+\infty} dt \cos(ut) [\Sigma_-^{(0)}(t) - \Sigma_+^{(0)}(t)] &= 2(2\sqrt{2}g) \end{aligned} \quad (\text{A.2})$$

and the ratios of hyperbolic functions admit a useful integral representation

$$\begin{aligned} t^{2k-1} \frac{\sinh \frac{t}{2\sqrt{2}g}}{\cosh \frac{t}{\sqrt{2}g}} &= (-1)^{k+1} g \int_{-\infty}^{+\infty} du \cos(ut) \frac{d^{2k-1}}{du^{2k-1}} \left[\frac{\sinh \frac{g\pi u}{\sqrt{2}}}{\cosh \sqrt{2}g\pi u} \right], \\ t^{2k-1} \frac{\cosh \frac{t}{2\sqrt{2}g}}{\cosh \frac{t}{\sqrt{2}g}} &= (-1)^k g \int_{-\infty}^{+\infty} du \sin(ut) \frac{d^{2k-1}}{du^{2k-1}} \left[\frac{\cosh \frac{g\pi u}{\sqrt{2}}}{\cosh \sqrt{2}g\pi u} \right], \\ t^{2k} \frac{\sinh \frac{t}{2\sqrt{2}g}}{\cosh \frac{t}{\sqrt{2}g}} &= (-1)^k g \int_{-\infty}^{+\infty} du \sin(ut) \frac{d^{2k}}{du^{2k}} \left[\frac{\sinh \frac{g\pi u}{\sqrt{2}}}{\cosh \sqrt{2}g\pi u} \right], \\ t^{2k} \frac{\cosh \frac{t}{2\sqrt{2}g}}{\cosh \frac{t}{\sqrt{2}g}} &= (-1)^k g \int_{-\infty}^{+\infty} du \cos(ut) \frac{d^{2k}}{du^{2k}} \left[\frac{\cosh \frac{g\pi u}{\sqrt{2}}}{\cosh \sqrt{2}g\pi u} \right]. \end{aligned}$$

Plugging them in the integrals \mathcal{B}_{2k-1} , \mathcal{C}_{2k} we obtain

$$\begin{aligned} \mathcal{B}_{2k-1}(g) &= (-1)^k g \left(\frac{1}{\sqrt{2}g}\right)^{2k} \int_{-\infty}^{+\infty} du \left[\int_0^{+\infty} dt \cos(ut) \frac{d^{2k-1}}{du^{2k-1}} \left[\frac{\sinh \frac{g\pi u}{\sqrt{2}}}{\cosh \sqrt{2}g\pi u} \right] (\Sigma_-^{(0)}(t) - \Sigma_+^{(0)}(t)) + \right. \\ &\quad \left. + \int_0^{+\infty} dt \sin(ut) \frac{d^{2k-1}}{du^{2k-1}} \left[\frac{\cosh \frac{g\pi u}{\sqrt{2}}}{\cosh \sqrt{2}g\pi u} \right] (\Sigma_-^{(0)}(t) + \Sigma_+^{(0)}(t)) \right], \\ \mathcal{C}_{2k}(g) &= (-1)^k g \left(\frac{1}{\sqrt{2}g}\right)^{2k+1} \int_{-\infty}^{+\infty} du \left[\int_0^{\infty} dt \cos(ut) \frac{d^{2k}}{du^{2k}} \left[\frac{\cosh \frac{g\pi u}{\sqrt{2}}}{\cosh \sqrt{2}g\pi u} \right] (\Sigma_-^{(0)}(t) - \Sigma_+^{(0)}(t)) + \right. \\ &\quad \left. + \int_0^{+\infty} dt \sin(ut) \frac{d^{2k}}{du^{2k}} \left[\frac{\sinh \frac{g\pi u}{\sqrt{2}}}{\cosh \sqrt{2}g\pi u} \right] (\Sigma_-^{(0)}(t) + \Sigma_+^{(0)}(t)) \right]. \end{aligned}$$

Let us evaluate them in the large g limit. The strategy is to split the integral over u in two intervals $|u| < 1$ and $|u| > 1$ in order to use the constraints (A.2). The former gives, together with the use of such constraints:

$$\begin{aligned}
& \int_{-1}^1 du \frac{d^{2k-1}}{du^{2k-1}} \left[\frac{\sinh \frac{g\pi u}{\sqrt{2}}}{\cosh \sqrt{2}g\pi u} \right] \int_0^{+\infty} dt \cos(ut) (\Sigma_-^{(0)}(t) - \Sigma_+^{(0)}(t)) = \\
&= 2(2\sqrt{2}g) \int_{-1}^1 du \frac{d^{2k-1}}{du^{2k-1}} \left[\frac{\sinh \frac{g\pi u}{\sqrt{2}}}{\cosh \sqrt{2}g\pi u} \right], \\
& \int_{-1}^1 du \frac{d^{2k-1}}{du^{2k-1}} \left[\frac{\cosh \frac{g\pi u}{\sqrt{2}}}{\cosh \sqrt{2}g\pi u} \right] \int_0^{+\infty} dt \sin(ut) (\Sigma_-^{(0)}(t) + \Sigma_+^{(0)}(t)) = 0, \\
& \int_{-1}^1 du \frac{d^{2k}}{du^{2k}} \left[\frac{\cosh \frac{g\pi u}{\sqrt{2}}}{\cosh \sqrt{2}g\pi u} \right] \int_0^{+\infty} dt \cos(ut) (\Sigma_-^{(0)}(t) - \Sigma_+^{(0)}(t)) = \\
&= 2(2\sqrt{2}g) \int_{-1}^1 du \frac{d^{2k}}{du^{2k}} \left[\frac{\cosh \frac{g\pi u}{\sqrt{2}}}{\cosh \sqrt{2}g\pi u} \right], \\
& \int_{-1}^1 du \frac{d^{2k}}{du^{2k}} \left[\frac{\sinh \frac{g\pi u}{\sqrt{2}}}{\cosh \sqrt{2}g\pi u} \right] \int_0^{+\infty} dt \sin(ut) (\Sigma_-^{(0)}(t) + \Sigma_+^{(0)}(t)) = 0.
\end{aligned}$$

Since we are interested in the large g behaviour, we can perform the previous integrals by rewriting them as the difference of the integrals with support over $(-\infty, +\infty)$ and $(-\infty, -1)$, $(1, +\infty)$, and finally taking the leading exponential in the integrands, so we will have for $n > 0$ (we will use a single index n because at this order there is no distinction between even and odd indexes):

$$-4(2\sqrt{2}g) \int_1^{+\infty} du \frac{d^n}{du^n} e^{-\frac{g\pi u}{\sqrt{2}}} = 8\sqrt{2}g \left(-\frac{\pi g}{\sqrt{2}} \right)^{n-1} e^{-\frac{g\pi}{\sqrt{2}}} + O(e^{-3\frac{g\pi}{\sqrt{2}}}).$$

The case with $n = 0$ needs to be treated separately, because we also have to take into account the contribution of the integral over $(-\infty, +\infty)$:

$$\int_{-1}^1 du \left[\frac{\cosh \frac{g\pi u}{\sqrt{2}}}{\cosh \sqrt{2}g\pi u} \right] = \int_{-\infty}^{+\infty} du \left[\frac{\cosh \frac{g\pi u}{\sqrt{2}}}{\cosh \sqrt{2}g\pi u} \right] - 2 \int_1^{+\infty} du \left[\frac{\cosh \frac{g\pi u}{\sqrt{2}}}{\cosh \sqrt{2}g\pi u} \right].$$

We end up with

$$2(2\sqrt{2}g) \int_{-1}^1 du \left[\frac{\cosh \frac{g\pi u}{\sqrt{2}}}{\cosh \sqrt{2}g\pi u} \right] = 4\sqrt{2} - 8\sqrt{2}g \left(\frac{\sqrt{2}}{g\pi} \right) e^{-\frac{g\pi}{\sqrt{2}}} + O(e^{-3\frac{g\pi}{\sqrt{2}}}). \quad (\text{A.3})$$

We stress that the above integrals have the same structure for any n only at leading order for large g . Next to leading orders will differ because of the different form of the integrands.

It is possible to show that, taking the leading exponential only, the integrals over $|u| > 1$ take the form

$$2 \left(-\frac{\pi g}{\sqrt{2}} \right)^n \int_1^{+\infty} du e^{-\frac{g\pi u}{\sqrt{2}}} \left[\int_0^{+\infty} dt \cos(ut) (\Sigma_-^{(0)}(t) - \Sigma_+^{(0)}(t)) + \int_0^{+\infty} dt \sin(ut) (\Sigma_-^{(0)}(t) + \Sigma_+^{(0)}(t)) \right],$$

which is accurate up to $O(e^{-3\frac{g\pi}{\sqrt{2}}})$ terms. The integral was estimated in [16], and taking into account the difference with the notations of that paper¹⁰, we have

$$\begin{aligned} & \int_1^{+\infty} du e^{-\frac{g\pi u}{\sqrt{2}}} \left[\int_0^{+\infty} dt \cos(ut) (\Sigma_-^{(0)}(t) - \Sigma_+^{(0)}(t)) + \int_0^{+\infty} dt \sin(ut) (\Sigma_-^{(0)}(t) + \Sigma_+^{(0)}(t)) \right] = \\ & = -\frac{\pi}{\sqrt{2}} m(g) + \frac{8}{\pi} e^{-\frac{g\pi}{\sqrt{2}}}, \end{aligned} \quad (\text{A.5})$$

where we pose, as before,

$$m(g) = k g^{1/4} e^{-\frac{\pi}{\sqrt{2}}g} [1 + O(1/g)], \quad k = \frac{2^{5/8} \pi^{1/4}}{\Gamma(5/4)}. \quad (\text{A.6})$$

It is interesting to notice that the $O(1/g)$ in the previous equation stands for power-like corrections which in principle can be computed by inspecting the sub-leading terms of the lhs of (A.5).

If we put everything together we have, up to the level of accuracy discussed before

$$\begin{aligned} \mathcal{B}_{2k-1}(g) &= (-1)^k \left[\frac{16\sqrt{2}}{\pi^2} \left(\frac{\pi}{2}\right)^{2k} e^{-\frac{g\pi}{\sqrt{2}}} - \frac{2\sqrt{2}}{\pi} \left(\frac{\pi}{2}\right)^{2k} \left(-\frac{\pi}{\sqrt{2}} m(g) + \frac{8}{\pi} e^{-\frac{g\pi}{\sqrt{2}}}\right) \right], \\ \mathcal{C}_{2k}(g) &= (-1)^k \left[-4\sqrt{2} \left(\frac{\pi}{2}\right)^{2k-1} e^{-\frac{g\pi}{\sqrt{2}}} + \sqrt{2} \left(\frac{\pi}{2}\right)^{2k} \left(-\frac{\pi}{\sqrt{2}} m(g) + \frac{8}{\pi} e^{-\frac{g\pi}{\sqrt{2}}}\right) \right], \\ \mathcal{C}_0(g) &= -\frac{8\sqrt{2}}{\pi} e^{-\frac{g\pi}{\sqrt{2}}} + 4 + \sqrt{2} \left(-\frac{\pi}{\sqrt{2}} m(g) + \frac{8}{\pi} e^{-\frac{g\pi}{\sqrt{2}}}\right). \end{aligned}$$

We notice that the first and the last terms always cancel, hence we have eventually:

$$\begin{aligned} \mathcal{B}_{2k-1}(g) &= (-1)^k \left(\frac{\pi}{2}\right)^{2k} 2m(g) + O(e^{-3\frac{g\pi}{\sqrt{2}}}), \\ \mathcal{C}_{2k}(g) &= (-1)^{k+1} \left(\frac{\pi}{2}\right)^{2k} \pi m(g) + O(e^{-3\frac{g\pi}{\sqrt{2}}}), \quad k > 0, \\ \mathcal{C}_0(g) &= 4 - \pi m(g) + O(e^{-3\frac{g\pi}{\sqrt{2}}}), = \sigma_H^{(0);(0)}. \end{aligned}$$

Therefore, we end up with the following estimates at large g (i.e., up to the order $O(e^{-3\frac{g\pi}{\sqrt{2}}})$):

$$\begin{aligned} \sqrt{2}g\tilde{S}_1^{(k)}(g) &= -\frac{1}{4\pi}\mathcal{B}_{2k-1}(g) = \frac{(-1)^{k+1}}{4\pi} \left(\frac{\pi}{2}\right)^{2k} 2m(g), \\ \sigma_H^{(0);(2k)} &= i^{-2k}\mathcal{C}_{2k}(g) = -\left(\frac{\pi}{2}\right)^{2k} \pi m(g), \quad k > 0, \\ \sigma_H^{(0);(0)} &= \mathcal{C}_0(g) = 4 - \pi m(g). \end{aligned} \quad (\text{A.7})$$

¹⁰It is easy to work out the following relations between our g , $\Sigma_{\pm}^{(0)}$ and the same quantities g_{BK} , $\Gamma_{\pm}^{(0)}$ in the paper [16]

$$g_{BK} = \frac{g}{\sqrt{2}}, \quad \Gamma_{\pm}^{(0)} = \frac{\Sigma_{\pm}^{(0)}}{2\sqrt{2}g}. \quad (\text{A.4})$$

We conclude with few words on the coefficient relevant for the computation of $f_1(g)$. Starting from (4.35) and using the BKK transformation (A.1), we get

$$\begin{aligned} \sqrt{2}gS_1^{(1)}(g) = & - \int_0^{+\infty} \frac{dt}{2t} \left[\frac{\sinh \frac{t}{2\sqrt{2}g}}{\cosh \frac{t}{\sqrt{2}g}} \left(\Sigma_-^{(0)}(t) - \Sigma_+^{(0)}(t) \right) + \right. \\ & \left. + \left(1 - \frac{\cosh \frac{t}{2\sqrt{2}g}}{\cosh \frac{t}{\sqrt{2}g}} \right) \left(\Sigma_-^{(0)}(t) + \Sigma_+^{(0)}(t) \right) \right], \end{aligned} \quad (\text{A.8})$$

which, after the use of the map (A.4), coincides with (53) of [16]. Therefore, the strong coupling analysis can be done following that paper.

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